HEEGAARD FLOER HOMOLOGY AND MORSE SURGERY

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Abstract. We establish surgery formulas for the filtration of the Heegaard Floer complex associated with $\frac{p}{q}$-surgery $Y_{\frac{p}{q}}(K)$ on a null-homologous knot (Y, K), induced by the core of the attached solid torus (which produces the surgery). This would generalize the result of Ozsváth and Szabó in [OS4]. We will also re-prove that surgery on non-trivial knots can not produce S^3, as a corollary of non-vanishing results for $\widehat{\mathrm{HFK}}(K_{\frac{p}{q}})$ where K is a knot in S^3.

1. Introduction

Suppose that K is a null-homologous knot in the three-manifold Y and suppose that $\frac{p}{q}$ is a positive rational number. One may consider a tubular neighborhood $\mathrm{nd}(K)$ of K in Y which may be identified with $S^1 \times D^2$ in such a way that the curve $S^1 \times \{1\}$ on the boundary of this solid torus has zero linking number with K in Y (thus has trivial image in the first homology of $Y \setminus \mathrm{nd}(K)$). Denote this curve by λ, and denote the meridian of K- which corresponds to $\{1\} \times \partial(D^2)$- by μ. Clearly the closed curve μ bounds a disk in Y. Replacing $\mathrm{nd}(K)$ with another solid torus such that the curve $p\mu + q\lambda$ on the boundary of the solid torus bounds a disk in the new solid torus produces the $\frac{p}{q}$-surgery on the knot K. We denote the resulting three-manifold by $Y_{\frac{p}{q}}(K)$. The central circle of this solid torus is a simple closed curve which would give a rationally null-homologous knot in $Y_{\frac{p}{q}}(K)$ denoted by $(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}})$ or just by $K_{\frac{p}{q}}$ in this paper.

The Heegaard Floer homology of $Y_{\frac{p}{q}}(K)$ is computed in [OS4] in terms of the Heegaard Floer complex associated with the knot (Y, K). It is the goal of this paper to understand the filtration of the complex given in [OS4] induced by the knot $K_{\frac{p}{q}}$. This is a special case of the question raised in [Ef1]. It will be used in [Ef2] to study the general case where two knot-complements are glued along their torus boundary.

In order to state the results obtained in this paper, let us introduce some notation. For simplicity assume that Y is a homology sphere. Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ be a pointed Heegaard diagram for the knot (Y, K), and let the corresponding Heegaard complex be generated by $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$. Assume that the complex $\mathrm{CFK}^\infty(Y, K)$ is equipped with the differential ∂^∞. For any intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $i(\mathbf{x}) \in \mathbb{Z} \cong \underline{\mathrm{Spin}}^c(Y, K)$ denote the associated relative Spin^c structure.

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Construct a complex $\mathbb{D}$ using generators $[\mathbf{x},i,j,k,l]\in(\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times\mathbb{Z}^4$ equipped with a differential $\partial_{\mathbb{D}}$ which is defined by

$$\partial_{\mathbb{D}}[\mathbf{x},i,j,k,l]=\sum_p n_p[\mathbf{y}_p,i-i_p,j-i_p,k-k_p,l-k_p],$$

whenever

$$\partial^\infty[\mathbf{x},i,k]=\sum_p n_p[\mathbf{y}_p,i-i_p,k-k_p].$$

For $a=[\mathbf{x},i,j,k,l]\in\mathbb{D}$ define $\Delta(a)=i-j+k-l$ and let $\mathbb{D}_\delta$ be the subcomplex of $\mathbb{D}$ generated by those $a\in\mathbb{D}$ as above such that $\Delta(a)=\delta$. Define $\mathbb{D}^{up}$ and $\mathbb{D}^{down}$ to be copies of the complex $\mathbb{D}_1$.

Define a $\mathbb{Z}\oplus\mathbb{Z}$ grading on the generators $\alpha=[\mathbf{x},i,j,k,l]$ of $\mathbb{D}^{up}\cup\mathbb{D}^{down}$ by

$$\mathcal{G}(\alpha)=(max(i,l),max(j,k)),\quad \text{if } \alpha\in\mathbb{D}^{up},$$
$$\mathcal{G}(\alpha)=(i,j)\quad \text{if } \alpha\in\mathbb{D}^{down}.$$

Suppose that u and w are points on the two sides of p which give the filtration on $\mathrm{CFK}^\infty(Y,K)$. The $\mathbb{Z}\oplus\mathbb{Z}$ filtration on $\mathbb{D}^{down}$ uses the first marked point u. If we denote by $\mathbb{D}^{down}_0$ the same chain complex with the $\mathbb{Z}\oplus\mathbb{Z}$ filtration coming from the last two integer components of the generators (corresponding to the marked point w) then there is a homotopy equivalence of filtered chain complexes

$$\tau:\mathbb{D}^{down}_0\longrightarrow\mathbb{D}^{down},$$

coming from the invariance of the chain homotopy type from the choice of the marked point.

Define the chain maps $g_{\frac{p}{q}}:\mathbb{D}^{up}\otimes\frac{\mathbb{Z}[\zeta]}{\zeta^q=1}\longrightarrow\mathbb{D}^{down}\otimes\frac{\mathbb{Z}[\zeta]}{\zeta^q=1}$ via the formula

$$g_{\frac{p}{q}}([\mathbf{x},i,j,k,l]\otimes\zeta^t)=\tau[\mathbf{x},l,k,2k-j-\lfloor\frac{t+p}{q}\rfloor,2l-i-\lfloor\frac{t+p}{q}\rfloor]\otimes\zeta^{t+p}.$$

Let $\overline{f}_{\frac{p}{q}}=Id+g_{\frac{p}{q}}$ be the sum of this map with the identity, and let $\mathbb{M}(\overline{f}_{\frac{p}{q}})$ denote the mapping cone of $\overline{f}_{\frac{p}{q}}$. The filtration $\mathcal{G}$ may naturally be extended to $\mathbb{M}(\overline{f}_{\frac{p}{q}})$.

The complex $\mathbb{M}(\overline{f}_{\frac{p}{q}})$ is decomposed as a direct sum according to the relative Spin^c structures. The Spin^c classes are assigned to the generators of $\mathbb{M}(\overline{f}_{\frac{p}{q}})$ via

$$\underline{\mathfrak{s}}([\mathbf{x},i,j,k,l]\otimes\zeta^t)=q(i(\mathbf{x})+j-k)+p(i-j)+t$$
$$\text{for } [\mathbf{x},i,j,k,l]\in\mathbb{D}^{up}\text{ or }\mathbb{D}^{down},\quad 0\le t<q.$$

Let P denote a domain in the lattice $\mathbb{Z}\oplus\mathbb{Z}$ such that for any $p=(i,j)\in P$ there exists some $N_p>0$ such that for any $i',j'>N_p$ the lattice point $(i-i',j-j')$ is not in P. Assume furthermore that if (i,j) and (i',j') are points in P then for any pair of integers $i''\in[i,i']$ and $j''\in[j,j']$ the point (i'',j'') is also included in P. Such a domain P will be called a *positive test domain.* If C is a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex with filtration $\mathcal{F}$, denote by C^P the module generated by $\mathcal{F}^{-1}(P)$, and equipped with the induced structure of a chain complex coming from C. We will denote by $H^P_*(C)$ the homology of the complex C^P. Two $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain

complexes C and C' are called *quasi-isomorphic* via a chain map $f : C \longrightarrow C'$ if for any positive test domain $P \subset \mathbb{Z} \oplus \mathbb{Z}$ the induced map in homology

$$f_*^P : H_*^P(C) \longrightarrow H_*^P(C')$$

is an isomorphism.

Among other things, the main theorem proved in this paper may be stated as follows:

Theorem 1.1. *Let Y be a homology sphere and let (Y, K) denote a knot in K. Suppose that $\frac{p}{q} > 0$ is a rational number and let $(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}})$, $\mathbb{D}^{up}$, $\mathbb{D}^{down}$ and $\mathbb{M}(\overline{f}_{\frac{p}{q}})$ be as before. Then for any relative* Spin^c *structure*

$$\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}}) = \underline{\mathrm{Spin}}^c(Y, K) \cong \mathbb{Z}$$

the knot Floer complex $\mathrm{CFK}^\infty(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}}, \underline{\mathfrak{t}})$ *is quasi-isomorphic to the complex* $\mathbb{M}(\overline{f}_{\frac{p}{q}})[\underline{\mathfrak{t}}]$.

In particular, when $P = \{(0, 0)\}$ the above theorem is used in the final section to prove the following non-vanishing result about $\widehat{\mathrm{HFK}}$ of rational surgeries on a knot:

Theorem 1.2. *Suppose that K is a knot in S^3 of genus $g(K)$, and let $r = \frac{p}{q} \in \mathbb{Q}$ be a positive rational number. Under the natural identification $\underline{\mathrm{Spin}}^c(S^3, K_r) = \mathbb{Z}$ we will have*

$$\widehat{\mathrm{HFK}}(K_r, -qg(K)) \cong \widehat{\mathrm{HFK}}(K_r, qg(K) + p - 1) \cong \widehat{\mathrm{HFK}}(K, g(K)) \neq 0,$$

and for any $\underline{\mathfrak{t}} \in \mathbb{Z}$ such that $\underline{\mathfrak{t}} < -qg(K)$ or $\underline{\mathfrak{t}} \geq qg(K)+p$ we will have $\widehat{\mathrm{HFK}}(K_r, \underline{\mathfrak{t}}) = 0$.

We may use this theorem to give an easy proof of a result of Gordon and Luecke ([GL]) than $\frac{1}{q}$-surgery on non-trivial knots in S^3 can note produce S^3. This is a special case of *Property P* proved by Kronheimer and Mrowka [KM]. In this form, it is also proved in [OS4] using Floer homology. However our proof is different from theirs.

Corollary 1.3. *If K is a knot in S^3 and $r = \frac{p}{q} \in \mathbb{Q}$ is a positive rational number such that the three-manifold obtained by r-surgery on K is S^3 then K is the unknot and $p = 1$.*

Proof. By considering the first Betti number it is clear that $p = 1$. If the three-manifold $S^3_r(K)$ is S^3, it is implied that $K_{\frac{1}{q}}$ is a knot L in S^3, and for any positive integer q', $K_{\frac{1}{q+q'}} = L_{\frac{1}{q'}}$. The previous non-vanishing theorem implies that $(q + q')g(K) = q'g(L)$ for any q', which implies that $g(K) = 0$. $\square$

We first consider the case of large integral surgeries in section 2. Together with a surgery exact sequence, this would suggest a computation of Heegaard Floer complex for arbitrary integral surgeries on null-homologous knots, which is done in section 3. In section 4 we will generalize this formula to the case of rationally null-homologous knots with minor modifications. In section 5 we combine these results to prove the above theorem for rational surgeries. In section 6 we study $\widehat{\mathrm{HFK}}(K_r)$ for positive rational numbers r and will prove the above non-vanishing result.

The formulas obtained here are essential for the study of the structure of Heegaard Floer complex that is associated with the closed three-manifold obtained by gluing two three-manifolds along their torus boundaries, which will appear in the sequel [Ef2].

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2. Heegaard Floer homology of large integral surgeries

The first step toward the desired computation is an understanding of the case of n surgeries, when n is a large integer. Suppose that (Y,K) is as above and consider a Heegaard diagram for the pair. Suppose that the curve β_g in the Heegaard diagram

$$H = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}, p)$$

corresponds to the meridian of K and that the marked point p is placed on β_g. One may assume that the curve β_g cuts α_g once and that this is the only element of $\boldsymbol{\alpha}$ that has an intersection point with β_g. Suppose that λ represents a longitude for the knot K (i.e. it cuts β_g once and stays disjoint from other elements of $\boldsymbol{\beta}$) such that the Heegaard diagram

$$(\Sigma, \boldsymbol{\alpha}, \{\beta_1, ..., \beta_{g-1}, \lambda\})$$

represents the three-manifold $Y_0(K)$. Winding λ around β_g- if it is done n times- would produce a Heegaard diagram for the three-manifold $Y_n(K)$. More precisely, if the resulting curve is denoted by λ_n, the Heegaard diagram

$$H_n = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n = \{\beta_1, ..., \beta_{g-1}, \lambda_n\}, p_n)$$

would give a diagram associated with the knot $(Y_n(K), K_n)$, where p_n is a marked point placed on λ_n.

Denote by u_n and w_n a pair of marked points on the two sides of λ_n which are both very close to p_n. In the description of the Heegaard complex associated with $Y_n(K)$ given in [OS1] we are interested in computing the $\mathbb{Z}\oplus\mathbb{Z}$-filtration induced by the pair of points (u_n, w_n).

Fix a Spin^c-structure $\mathfrak{s} \in \mathrm{Spin}^c(Y_n(K))$ and choose the marked point p_n so that all the generators of the complex $\mathrm{CF}^\infty(Y_n(K), \mathfrak{s})$ are supported in the winding region if the Spin^c classes are assigned using either of the marked points u_n or w_n on the two sides of λ_n.

The curve λ_n intersects the α-curve α_g in n-points which appear in the winding region (there may be other intersections outside the winding region). Denote these points of intersection by

$$..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...,$$

where x_1 is the intersection point with the property that three of its four neighboring quadrants belong to the regions that contain either u_n or w_n. Any generator which is supported in the winding region is of the form

$$\{x_i\} \cup \mathbf{y}_0 = \{y_1, ..., y_{g-1}, x_i\},$$

and it is in correspondence with the generator

$$\mathbf{y} = \{x\} \cup \mathbf{y}_0 = \{y_1, ..., y_{g-1}, x\}$$

for the complex associated with the knot (Y, K), where x denotes the unique intersection point of α_g and β_g. Denote the former generator by $(\mathbf{y})_i$, keeping track of the intersection point x_i among those in the winding region.

Note that $\underline{\mathrm{Spin}}^c(Y, K) = \mathrm{Spin}^c(Y_0(K)) = \mathbb{Z} \oplus \mathrm{Spin}^c(Y)$ and for $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(Y, K)$ let $i(\underline{\mathfrak{s}})$ denote the first component in this decomposition. Remember that there is a map

$$\underline{\mathfrak{s}} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\mathrm{Spin}}^c(Y, K),$$

which is defined in [OS1] or more generally in [OS4]. Thus to any generator $\mathbf{y}$ as above we may associate an integer $i(\underline{\mathfrak{s}}(\mathbf{y})) \in \mathbb{Z}$.

Similarly, $\mathrm{Spin}^c(Y_n(K)) = \frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \mathrm{Spin}^c(Y)$ and for $\mathfrak{s}_n \in \mathrm{Spin}^c(Y_n(K))$ we may denote the projection over the first component of this decomposition by $i_n(\mathfrak{s}_n)) \in \frac{\mathbb{Z}}{n\mathbb{Z}}$. The marked point gives a map

$$\mathfrak{s}_n : \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_n} \longrightarrow \mathrm{Spin}^c(Y_n(K)).$$

As a result, for any generator $\mathbf{y}$ as above and any integer $i \in \mathbb{Z}$ which is not *very large* we obtain a number

$$i_n(\mathfrak{s}_n((\mathbf{y})_i)) = [i_n(\mathfrak{s}_n((\mathbf{y})_0)) - i] \in \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

Note that there is a relation between these numbers given by the following lemma which is in fact proved in [OS1]:

Lemma 2.1. *With the above notation, for any generator* $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *we have*

$$i(\underline{\mathfrak{s}}(\mathbf{y})) - i = i_n(\mathfrak{s}_n((\mathbf{y})_i)) \pmod{n}.$$

Let $s : \underline{\mathrm{Spin}}^c(Y, K) \longrightarrow \mathrm{Spin}^c(Y)$ and $s_n : \underline{\mathrm{Spin}}^c(Y_n(K), K_n) \longrightarrow \mathrm{Spin}^c(Y_n(K))$ denote the natural maps obtained by extending the relative Spin^c structures of the knot complements over the attached solid torus.

For a fixed Spin^c-structure $\mathfrak{t}_n \in \mathrm{Spin}^c(Y_n(K))$ define

$$\mathrm{CFK}^\infty(Y, K, \mathfrak{t}_n) :=$$

$$\Big\langle \Big\{ [\mathbf{y}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z} \;\Big|\; \begin{array}{c} s_n(\underline{\mathfrak{s}}(\mathbf{y}) - (i-j)\mathrm{PD}[\mu]) = \mathfrak{t}_n \\ -\frac{n}{2} \leq i(\underline{\mathfrak{s}}(\mathbf{y})) - i + j < \frac{n}{2} \end{array} \Big\} \Big\rangle,$$

where μ denotes the image of the meridian of the knot K in the three-manifold $Y_0(K)$ obtained by a zero-surgery on K.

It is shown in [OS1] that for large values of n the complex $\mathrm{CFK}^\infty(Y,K,\mathfrak{t}_n)$ may in fact be thought of as giving the complex $\mathrm{CF}^\infty(Y_n(K),\mathfrak{t}_n)$ (as a $\mathbb{Z}$-filtered chain complex) under a correspondence which may be naively described as

$$[\mathbf{y},i,j]\mapsto [(\mathbf{y})_{i-j},\ \max\{i,j\}].$$

This map is constructed by counting holomorphic triangles corresponding to the triple Heegaard diagram

$$R_n=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_n;u',w').$$

Here $\boldsymbol{\beta}_n=\{\beta_1',...,\beta_{g-1}',\lambda_n\}$ denotes a set of g simple closed curves such that the first $g-1$ of them are in fact close isotopic copies of the curves in $\boldsymbol{\beta}$ with the property that each β_i' cuts the corresponding curve β_i in a pair of cancelling intersection points, and u',w' are two marked points on the two sides of the curve β_g very close to the point $p\in\beta_g$. The curve β_g is chosen so that it is located almost in the middle of the winding region. We may abuse the notation and write

$$\boldsymbol{\beta}_n=\{\beta_1,...,\beta_{g-1},\lambda_n\}.$$

The inverse of the above map is in fact of the form

$$[\mathbf{y},i,j]\mapsto [(\mathbf{y})_{i-j},\ \max\{i,j\}]+\text{ lower order terms},$$

which would then imply that the structure of a filtered chain complex induced on the right-hand-side using the initial map $[\mathbf{y},i,j]\mapsto [(\mathbf{y})_{i-j},\ \min\{i,j\}]$ is the same as its own structure as a filtered chain complex, see [OS1] for a more detailed description.

Choose the curve β_g so that it cuts λ_n exactly once in $\{y\}=\beta_g\cap\lambda_n$ and the intersection point $\{x\}=\alpha_g\cap\beta_g$ is located between x_0 and x_1. Furthermore, there is a small triangle Δ_0 with vertices x,x_0,y on the surface Σ, and another one- denoted by Δ_1- with vertices x,x_1,y. There are 4 quadrants around y, two of them being parts of the triangles Δ_0 and Δ_1. Denote the remaining two regions by D_1 and D_2, so that D_1 is on the right-hand-side of both β_g and λ_n and so that D_2 is on the left-hand-side of both of them. Let u,v,w and z be four marked points in D_1,Δ_1,D_2 and Δ_0 respectively (see figure 1).

Denote the new Heegaard diagram by

$$S_n=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n,\boldsymbol{\beta};u,v,w,z).$$

There is a holomorphic triangle map associated with S_n which we will study below. Note that the Heegaard diagram

$$\overline{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v,w,z)$$

corresponds to a chain complex $\mathbb{D}$. The generators of $\mathbb{D}$ are of the form

$$[\mathbf{x},i,j,k,l]\in(\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}=(\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times\mathbb{Z}^4.$$

If ∂^∞ denotes the boundary map associated with the filtered chain complex $\mathrm{CFK}^\infty(Y,K)$ assigned to the Heegaard diagram H for the knot and

$$\partial^\infty[\mathbf{x},i,k]=\sum_p n_p[\mathbf{y}_p,i-i_p,k-k_p],$$

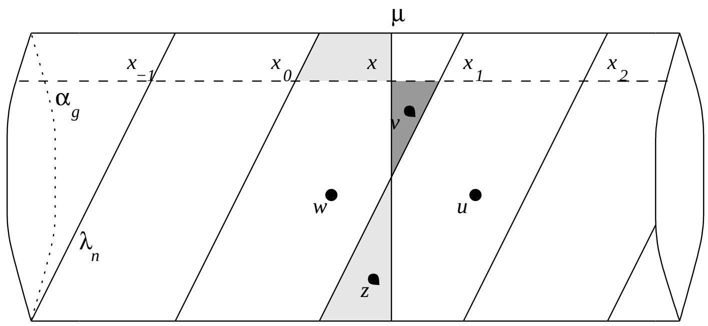

FIGURE 1. The Heegaard diagram S_n. The shaded triangles are Δ_0 and Δ_1.

then the differential $\partial_{\mathbb{D}}$ for $\mathbb{D}$ is defined by

$$\partial_{\mathbb{D}}[\mathbf{x},i,j,k,l]=\sum_p n_p[\mathbf{y}_p,i-i_p,j-i_p,k-k_p,l-k_p].$$

We may note that if two generators $[\mathbf{x},i,j,k,l]$ and $[\mathbf{y},i',j',k',l']$ are connected by a topological disk $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ then we will have the identities

$$\begin{cases}\mathfrak{s}(\mathbf{x})=\mathfrak{s}(\mathbf{y})\\ i(\underline{\mathfrak{s}}(\mathbf{x}))-(i-k)=i(\underline{\mathfrak{s}}(\mathbf{y}))-(i'-k')\\ i-j=i'-j',\quad k-l=k'-l',\end{cases}\tag{1}$$

where we define $\mathfrak{s}(\mathbf{x})=s(\underline{\mathfrak{s}}(\mathbf{x}))$.

In $\mathbb{D}$ let $\mathbb{D}_1$ denote the subcomplex generated by all $a=[\mathbf{x},i,j,k,l]$ such that $\Delta(a)=i-j+k-l$ is equal to 1. The triangle map Φ maps a third complex $\mathbb{E}_0\subset\mathbb{E}$ to the complex $\mathbb{D}_1$. Here $\mathbb{E}$ is the filtered chain complex associated with the Heegaard diagram

$$H''=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v,w,z),$$

which is a Heegaard diagram associated to $\mathrm{CFK}^\infty(Y_n(K),K_n)$ in a way similar to the above correspondence between $\mathbb{D}$ and $\mathrm{CFK}^\infty(Y,K)$. Here the two marked points w and v are in the same domain and similarly u and z are in the same domain (the last condition above is thus replaced by $j-k=j'-k'$ and $i-l=i'-l'$). Denote by $\mathbb{E}_0$ the subcomplex of $\mathbb{E}$ consisting of the tuples

$$\mathbb{E}_0=\Big\langle\Big\{[\mathbf{x},i,j,k,l]\in\mathbb{E}\;\Big|\;\begin{matrix}j-k=0\\ i-l=0\end{matrix}\Big\}\Big\rangle.$$

The complex $\mathbb{E}_0$ may be identified with the chain complex $\mathrm{CFK}^\infty(Y_n(K),K_n)$. We will sometimes abuse the notation and denote the element $[\mathbf{x},i,j,j,i]\in\mathbb{E}_0$ by $[\mathbf{x},i,j]$.

The triangle map $\Phi:\mathbb{E}\longrightarrow\mathbb{D}$ reduces to a map $\Phi_0:\mathbb{E}_0\longrightarrow\mathbb{D}_1$. This may be checked by examining the local multiplicities around the intersection of $\mu=\beta_g$ and λ_n.

In terms of the Energy filtration and the $\mathbb{Z}\oplus\mathbb{Z}$-filtration, the map Φ_0 is defined so that:

$$\Phi_0^{-1}[\mathbf{x},i,j,k,l] = [(\mathbf{x})_{k-j}, \max\{i,l\},\ \max\{j,k\}] + \text{terms of lower order},$$

when $|k-j|$ is small. Having this in mind, equip $\mathbb{D}_1$ with a $\mathbb{Z}\oplus\mathbb{Z}$ filtration

$$\mathcal{F}[\mathbf{x},i,j,k,l] = (\max\{i,l\},\ \max\{j,k\}).$$

The set of relative Spin^c structures for the knot $(Y_n(K),K_n)$ is easy to understand, according to [OS4]:

$$\begin{aligned}\underline{\mathrm{Spin}}^c(Y_n(K),K_n) &= \underline{\mathrm{Spin}}^c(Y_n(K)\setminus \mathrm{nd}(K_n), \partial(Y_n(K)\setminus \mathrm{nd}(K_n)))\\ &= \underline{\mathrm{Spin}}^c(Y\setminus \mathrm{nd}(K), \partial(Y\setminus \mathrm{nd}(K)))\\ &= \underline{\mathrm{Spin}}^c(Y,K) = \mathrm{Spin}^c(Y)\oplus\mathbb{Z}\end{aligned} \tag{2}$$

There is a map which projects relative Spin^c structures over Spin^c structures of $Y_n(K)$:

$$\begin{aligned} G_n = G_{Y_n(K),K_n}(=s_n) : \mathrm{Spin}^c(Y)\oplus\mathbb{Z} = \underline{\mathrm{Spin}}^c(Y_n(K),K_n)\\ \longrightarrow \mathrm{Spin}^c(Y_n(K)) = \mathrm{Spin}^c(Y)\oplus\frac{\mathbb{Z}}{n\mathbb{Z}}\end{aligned}$$

The map G_n is simply the reduction modulo the integer n, i.e. $G_n(\mathfrak{s},i) = (\mathfrak{s},(i)_{\text{mod } n})$. There is a relative Spin^c structure associated with any generator of the complex $\mathrm{CFK}^\infty(Y_n(K),K_n)$. The intersection point associated with $[\mathbf{x},i,j,k,l]$ is $(\mathbf{x})_{k-j}$ and the relative Spin^c structure associated with this generator is

$$\underline{\mathfrak{s}}(\mathbf{x}) + (j-k+n\epsilon(k-j))\mathrm{PD}[\mu] \in \underline{\mathrm{Spin}}^c(Y,K),$$

where $\epsilon(i)$ is equal to 1 if i is positive and is zero otherwise. This may be checked easily from lemma 2.1 in [OS1], at least in the relative version.

We may assign relative Spin^c structures to the generators of $\mathbb{D}_1$ so that the map Φ preserves the relative Spin^c class. With the above computation and the filtration $\mathcal{F}$ in mind, this assignment should be defined via

$$\begin{aligned} \underline{\mathfrak{s}} : (\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times\mathbb{Z}^4 \longrightarrow & \underline{\mathrm{Spin}}^c(Y_n(K),K_n) = \mathrm{Spin}^c(Y)\oplus\mathbb{Z}\\ \underline{\mathfrak{s}}[\mathbf{x},i,j,k,l] = & \underline{\mathfrak{s}}(\mathbf{x}) + (j-k+n\epsilon(k-j))\mathrm{PD}[\mu]\\ & + n(\max\{i,l\}-\max\{j,k\})\mathrm{PD}[\mu]\end{aligned}$$

Note that the last term is $n(i-j-\epsilon(k-j))$ since $i-j+k-l=1$. It is implied that

$$\underline{\mathfrak{s}}[\mathbf{x},i,j,k,l] = \underline{\mathfrak{s}}(\mathbf{x}) + (j-k+n(i-j))\mathrm{PD}[\mu].$$

The Spin^c structure in $\underline{\mathrm{Spin}}^c(Y_n(K))$ associated with a generator $a=[\mathbf{x},i,j,k,l]$ as above is

$$\mathfrak{s}_n(a) = (s(\mathbf{x}), (i(\underline{\mathfrak{s}}(\mathbf{x}))+j-k)_{\text{mod } n}) \in \mathrm{Spin}^c(Y)\oplus\frac{\mathbb{Z}}{n\mathbb{Z}} = \mathrm{Spin}^c(Y_n(K)).$$

For $s\in\mathbb{Z}$ define $\mathbb{D}_1^s$ to be the subcomplex of $\mathbb{D}_1$ generated by $[\mathbf{x},i,j,k,l]$ such that $i(\underline{\mathfrak{s}}(\mathbf{x}))+j-k=s$. Fix a relative Spin^c structure $\underline{\mathfrak{s}}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n) = \underline{\mathrm{Spin}}^c(Y,K)$ and let $-\frac{n}{2}\le s<\frac{n}{2}$ be an integer with the property that $i(\underline{\mathfrak{s}}) = s \pmod n$. Compose the map $\Phi : \mathrm{CFK}^\infty(Y_n(K),K_n,\underline{\mathfrak{s}})\to\mathbb{D}_1$ with the projection over $\mathbb{D}_1^s(\underline{\mathfrak{s}})$. Here $\mathbb{D}_1^s(\underline{\mathfrak{s}})$ is generated by those generators of $\mathbb{D}_1^s$ which are in the

relative Spin^c class $\underline{\mathfrak{s}}$, i.e. with $[\mathbf{x},i,j,k,l]$ such that $\underline{\mathfrak{s}}(\mathbf{x})+(j-k+n(i-j))\mathrm{PD}[\mu]=\underline{\mathfrak{s}}$. It is not hard to show that this induces a chain homotopy equivalence

$$\Phi^{\underline{\mathfrak{s}}}:\mathrm{CFK}^\infty(Y_n(K),K_n,\underline{\mathfrak{s}})\longrightarrow \mathbb{D}_1^s(\underline{\mathfrak{s}}).$$

We have proved the following theorem.

Theorem 2.2. *Suppose that (Y,K) is a null-homologous knot, n is a sufficiently large integer, and $(Y_n(K),K_n)$, the Heegaard diagram H and the complexes $\mathbb{D}_1$ and $\mathbb{D}_1^s(\underline{\mathfrak{s}})$ are as above. Then the knot Floer complex associated with the rationally null-homologous knot $(Y_n(K),K_n)$ in the relative* Spin^c *class $\underline{\mathfrak{s}}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n)$ has the same filtered chain homotopy type as the complex $\mathbb{D}_1^s(\underline{\mathfrak{s}})$ equipped with the $\mathbb{Z}\oplus\mathbb{Z}$ filtration given by*

$$\mathcal{F}[\mathbf{x},i,j,k,l]=(max\{i,l\},\ max\{j,k\}).$$

Here $-\frac{n}{2}\le s<\frac{n}{2}$ is chosen so that $i(\underline{\mathfrak{s}})=s\pmod n$ and the decomposition of $\mathbb{D}_1$ into a direct sum according to the relative Spin^c *structures is given by*

$$\mathbb{D}_1(\underline{\mathfrak{t}})=\Big\langle\Big\{[\mathbf{x},i,j,k,l]\in\mathbb{D}_1\ \Big|\ \underline{\mathfrak{s}}(\mathbf{x})+(j-k+n(i-j))\mathrm{PD}[\mu]=\underline{\mathfrak{t}}\Big\}\Big\rangle.$$

This computation finishes our study of the surgery formulas when the integer n is large. We will use this computation in the upcoming section to understand the chain complex associated with the knot $(Y_n(K),K_n)$, where n is an arbitrary non-zero integer.

Remark 2.3. *Note that the computation of Ozsváth and Szabó in* [OS1] *of the complex $\mathrm{CF}^+(Y_n(K))$ is in fact a corollary of the above theorem, once we write down the relation between the filtered chain complex $\mathrm{CFK}^\infty(Y_n(K),K_n)$ and the Heegaard Floer homology associated with the ambient three-manifold $Y_n(K)$ as introduced in* [OS4].

Remark 2.4. *If K is an alternating knot in S^3, the knot Floer complex associated with K may be computed as in* [Ras1, OS2]. *As a result the above argument gives a computation of the complexes $\mathrm{CFK}^\infty(S^3_n(K),K_n)$ where n is sufficiently large, which is not hard to do by hand once the symmetrized Alexander polynomial and the signature are given.*

3. Modification of Ozsváth-Szabó argument

We begin by reminding the reader of the argument given by Ozsváth and Szabó in [OS3] to generalize the computation of $\mathrm{CF}^+(Y_n(K))$ when n is a large integer to the case of arbitrary integer n.

In [OS3], the first step is the construction of an exact sequence coming from the quadruple Heegaard diagram

$$R_{m,n}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{m+n})$$

where $\boldsymbol{\beta},\boldsymbol{\beta}_n$ and $\boldsymbol{\beta}_{n+m}$ are as before, and each of them is equipped with a special curve denoted by $\mu=\beta_g,\lambda_n$ and λ_{m+n} respectively. Furthermore, we assume that the curves in $\boldsymbol{\beta}_n$ are small isotopic copies of the curves in $\boldsymbol{\beta}$ (except for λ_n which is not an isotopic copy of μ) such that there is a pair of cancelling intersection points between any curve in $\boldsymbol{\beta}_n$ and the corresponding element of $\boldsymbol{\beta}$. The same

assumption is made for $\boldsymbol{\beta}_{m+n}$ and the same is assumed for the relation between the curves in $\boldsymbol{\beta}_n$ and those in $\boldsymbol{\beta}_{m+n}$. A marked point p is fixed on μ and a marked point u on the diagram is also chosen. Using the marked points u and p three chain maps are constructed:

$$\begin{aligned} f_1^+ &: \mathrm{CF}^+(Y_n(K)) \longrightarrow \mathrm{CF}^+(Y_{m+n}(K)) \\ f_2^+ &: \mathrm{CF}^+(Y_{m+n}(K)) \longrightarrow \bigoplus^m \mathrm{CF}^+(Y) \\ f_3^+ &: \bigoplus^m \mathrm{CF}^+(Y) \longrightarrow \mathrm{CF}^+(Y_n(K)), \end{aligned}$$

where the map f_2^+ counts the number of times the boundary of a holomorphic triangle touches the point p, in order to give a map to the chain complex with twisted coefficients

$$\mathrm{CF}^+(Y, \mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}]) \cong \mathrm{CF}^+(Y) \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[T]}{T^m = 1}.$$

As the integer $m = \ell n$ becomes large the complex $\mathrm{CF}^+(Y_n(K))$ is proved to be chain homotopic to the limit of the mapping cones of the chain maps f_2^+ via the map induced by f_1^+. Since for large choices of ℓ the complex $\mathrm{CF}^+(Y_{(\ell+1)n}(K))$ is described in terms of $\mathrm{CFK}^\infty(Y, K)$ in [OS1], one would be done with the computation once the compatibility of certain maps and isomorphisms are verified, and the limiting behavior as m goes to infinity is studied.

In this section we will basically follow the same strategy using more marked points. Add the marked points u, v, w and z to the Heegaard diagram $R_{m,n}$ in the regions described bellow. Choose an intersection point between the curves λ_n and λ_{m+n} in the middle of the winding region, denoted by q. We will assume that $m = \ell n$ for an integer ℓ which is chosen to be appropriately large. We continue to assume that α_g is the unique α-curve in the winding region. From the 4 quadrants around the intersection point q, two of them are parts of small triangles Δ_0 and Δ_1 between $\boldsymbol{\alpha}, \boldsymbol{\beta}_n$ and $\boldsymbol{\beta}_{m+n}$. We may assume that the intersection points between α_g and λ_{m+n} in the winding region are

$$..., y_{-2}, y_{-1}, y_0, y_1, y_2, ...,$$

and that the intersection points between λ_n and α_g are

$$..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...$$

as before. We may also assume that the domain Δ_i for $i = 0, 1$ is the triangle with vertices q, x_i and y_i, and Δ_1 is one of the connected domains in the complement of curves $\Sigma \setminus C$ where

$$C = \boldsymbol{\alpha} \cup \boldsymbol{\beta}_n \cup \boldsymbol{\beta}_{m+n}.$$

Other that Δ_0 and Δ_1 there are two other domains which have q as a corner. One of them is on the right-hand-side of both λ_n and λ_{m+n}, denoted by D_1, and the other one is on the left-hand-side of both of them, denoted by D_2. The domains D_1 and D_2 are assumed to be connected regions in the complement of the curves $\Sigma \setminus C$. We may assume that the meridian μ passes through the regions D_1, D_2 and Δ_0, cutting each of them into two parts: $\Delta_0 = \Delta_0^R \cup \Delta_0^L$, $D_1 = D_1^R \cup D_1^L$ and $D_2 = D_2^R \cup D_2^L$. Here $\Delta_0^R \subset \Delta_0$ is the part on the right-hand-side of μ and Δ_0^L is the part on the left-hand-side. Similarly for the other partitions. Choose

the marked points so that u is in D_1^R, v is in D_2^R, w is in D_2^L and z is in D_1^L (see figure 2). We obtain the Heegaard diagram

$$R_{m,n} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{m+n}; u, v, w, z),$$

which will be used for constructing certain relevant chain complexes and chain maps between them.

We are interested in the mapping cone of the chain map f defined from the filtered chain complex $\mathbb{B}_{m+n}$ associated with the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}_{m+n})$ to the chain complex $\mathbb{B}$ associated with $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The map g going from the complex $\mathbb{B}_n$ associated with $(\boldsymbol{\alpha}, \boldsymbol{\beta}_n)$ to $\mathbb{B}_{m+n}$ defines a map $\overline{g}$ from $\mathbb{B}_n$ to the mapping cone $\mathbb{M}(f)$ of f, and the chain map h from $\mathbb{B}$ to $\mathbb{B}_n$ defines a map $\overline{h}$ from $\mathbb{M}(f)$ to $\mathbb{B}_n$. More precisely, the Heegaard diagram

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; (u, z), (v, w))$$

will produce the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex $\mathbb{B}_n$ which has the chain homotopy type of $\mathrm{CFK}^\infty(Y_n(K), K_n)$. Here by putting the pairs of points in parenthesis we mean that in the relevant Heegaard diagram the two points are in the same domain in the complement of the curves appearing in the Heegaard diagram. The triangle map which defines g is associated with the Heegaard triple

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{m+n}; (u, z), (v, w)),$$

and will provide a map from $\mathbb{B}_n$ to the filtered chain complex $\mathbb{B}_{m+n}$ (defined using the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{m+n}; (u, z), (v, w))$). A canonical generator in the complex associated with $(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{m+n}; u, w)$ is chosen as specified in [OS4], and is used to define g (this is a Heegaard diagram for the canonical knot $O_{\frac{m}{1}}$ of the Lens space $L(m, 1)$).

The Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ defines a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex with twisted coefficients, with the help of marked points w and z. More precisely, consider the chain complex $\mathbb{B}[m]$ generated by the generators $[\mathbf{x}, i, j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

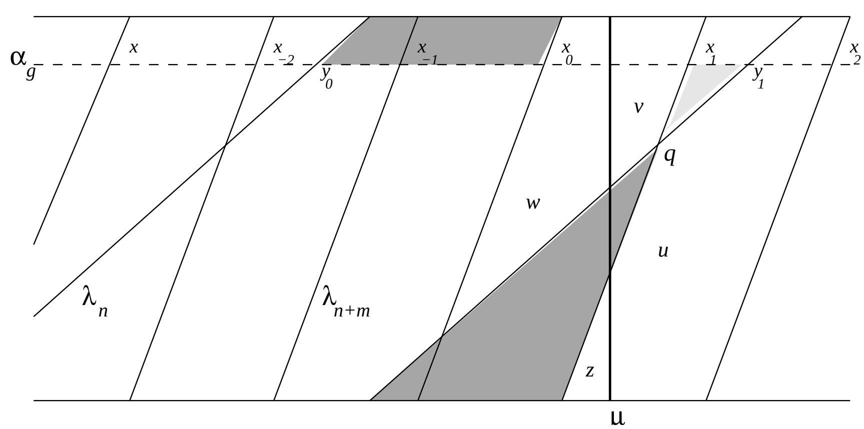

FIGURE 2. The Heegaard diagram $R_{m,n}$. The shaded triangles are Δ_0 and Δ_1. The marked points u, v, w and z are placed in D_1^R, D_2^R, D_2^L and D_1^L respectively.

and $i,j\in\mathbb{Z}$ over the coefficient ring $\mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}]=\frac{\mathbb{Z}[T]}{T^m=1}$. The boundary map is defined via

$$\partial^m[\mathbf{x},i,j]=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \phi\in\pi_2(\mathbf{x},\mathbf{y}),\ \mu(\phi)=1}}\#(\widehat{\mathcal{M}}(\phi))[\mathbf{y},i-n_u(\phi),j-n_v(\phi)]T^{m_p(\phi)},$$

where as usual $n_u(\phi)$ and $n_v(\phi)$ are the intersection numbers of the disk ϕ with the hyper-surfaces $\{u\}\times\mathrm{Sym}^{g-1}\Sigma$ and $\{v\}\times\mathrm{Sym}^{g-1}\Sigma$ respectively. Here $m_p(\phi)=n_w(\phi)-n_v(\phi)$ is the intersection number of $\partial\phi$ with the submanifold

$$\beta_1\times\beta_2\times\ldots\times\beta_{g-1}\times\{p\}\subset\mathbb{T}_\beta$$

of $\mathbb{T}_\beta$. The point p is chosen to be a marked point on $\mu=\beta_g$ on the arc between D_2^R and D_2^L. Note that $n_u(\phi)=n_v(\phi)$ and that $n_w(\phi)-n_v(\phi)=n_z(\phi)-n_u(\phi)$. The complex $\mathbb{B}[m]$ may be constructed from the knot Floer complex $\mathrm{CFK}^\infty(Y,K)$ as the above equations suggest.

Suppose that $\mathbb{B}$ denotes the complex generated over $\mathbb{Z}$ by $[\mathbf{x},i,j]$ such that $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $i,j\in\mathbb{Z}$, equipped with the differential

$$\partial[\mathbf{x},i,j]=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}}\#(\widehat{\mathcal{M}}(\phi))[\mathbf{y},i-n_u(\phi),j-n_v(\phi)].$$

This complex may in fact be constructed out of $\mathrm{CF}^\infty(Y,K)$. Then there is a chain map

$$\theta:\mathbb{B}[m]\longrightarrow\mathbb{B}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}]$$

defined by $\theta([\mathbf{x},i,j]T^k)=[\mathbf{x},i,j]\otimes T^{k+m_p(\phi_\mathbf{x})}$, where $\phi_\mathbf{x}\in\pi_2(\mathbf{x},\mathbf{x}_0)$ is an arbitrary disk for a fixed intersection point $\mathbf{x}_0$ of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in the Spin^c class of $\mathbf{x}$. Note that $m_p(\phi_\mathbf{x})$ depends only on $\mathbf{x},\mathbf{x}_0$ and is independent of the choice of $\phi_\mathbf{x}\in\pi_2(\mathbf{x},\mathbf{x}_0)$.

The Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{m+n},\boldsymbol{\beta};u,v,w,z)$ produces a pair of chain map f_v and f_w from the complex $\mathbb{B}_{m+n}$ to $\mathbb{B}[m]$, defined as follows. The Heegaard diagram $(\Sigma,\boldsymbol{\beta}_{m+n},\boldsymbol{\beta},(u,v),(w,z))$ is a Heegaard diagram for the trivial knot in $\#^{g-1}(S^1\times S^2)$. Denote the canonical generator by $[\Theta_0,0,0]$. Pairing Θ_0 with the generators of $\mathbb{B}_{m+n}$, using the triangle map associated with the above Heegaard diagram and the marked points u,v, and twisting according to the difference between the intersection numbers at v and w we obtain a chain map f_v from $\mathbb{B}_{m+n}$ to $\mathbb{B}[m]$ which is naively defined via

$$f_v[\mathbf{x},i,j]=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \psi\in\pi_2[\mathbf{x},\Theta_0,\mathbf{y}]\\ \mu(\psi)=0}}\#(\mathcal{M}(\psi))[\mathbf{y},i-n_u(\psi),j-n_v(\psi)]T^{m_w(\psi)-n_v(\psi)}.$$

The second map f_w is defined similarly:

$$f_w[\mathbf{x},i,j]=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \psi\in\pi_2[\mathbf{x},\Theta_0,\mathbf{y}]\\ \mu(\psi)=0}}\#(\mathcal{M}(\psi))[\mathbf{y},i-n_z(\psi),j-n_w(\psi)]T^{m_w(\psi)-n_v(\psi)}.$$

Suppose that the marked point $p \in \mu$ is chosen so that it is located on the arc between D_2^R and D_2^L. The map $m_p : \pi_2(\mathbf{x}, \Theta_0, \mathbf{y}) \longrightarrow \frac{\mathbb{Z}}{m\mathbb{Z}}$ lifts to a function

$$\mathfrak{m} : \mathrm{Spin}^c(W_{m+n}(K)) \longrightarrow \mathbb{Z},$$

where $W_{m+n}(K) : Y_{m+n}(K) \longrightarrow Y$ is the 4-manifold cobordism between these two three-manifolds which is given by our Heegaard diagram. If $\widehat{F} \subset W_{m+n}(K)$ is the surface obtained by capping a Seifert surface associated with the knot K in $W_{m+n}(K)$, the map $\mathfrak{m}$ satisfies the relation

$$\mathfrak{m}(\mathfrak{t} - \mathrm{PD}[\widehat{F}]) = \mathfrak{m}(\mathfrak{t}) + m + n.$$

Thus the map $\theta \circ f_v$ may be written as

$$\theta \circ f_v(a) = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(W_{m+n}(K))} f_v^{\mathfrak{t}}(a) \otimes T^{\mathfrak{m}(\mathfrak{t})},$$

where $f_v^{\mathfrak{t}}$ corresponds to counting holomorphic triangles which induce the Spin^c structure $\underline{\mathfrak{t}}$ on the cobordism. Similarly we have

$$\theta \circ f_w(a) = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(W_{m+n}(K))} f_w^{\mathfrak{t}}(a) \otimes T^{\mathfrak{m}(\mathfrak{t})}.$$

Fix a Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y_{m+n}(K))$ and let $-\frac{m+n}{2} \leq s < \frac{m+n}{2}$ be an integer such that its class modulo $m+n$ represents $i_{m+n}(\mathfrak{t}) \in \frac{\mathbb{Z}}{(m+n)\mathbb{Z}}$. Denote by $\mathfrak{x}_{\mathfrak{t}}$ and $\mathfrak{y}_{\mathfrak{t}}$ the Spin^c-structures on $W_{m+n}(K)$ such that they restrict to $\mathfrak{t}$ on $Y_{m+n}(K)$ and

$$\langle c_1(\mathfrak{x}_{\mathfrak{t}}), \mathrm{PD}[\widehat{F}]\rangle + n + m = 2s \quad \text{and} \quad \langle c_1(\mathfrak{y}_{\mathfrak{t}}), \mathrm{PD}[\widehat{F}]\rangle - n - m = 2s.$$

Since for $m = n\ell$ we have $[\widehat{F}].[\widehat{F}] = -n(\ell+1)$, we obtain the relation $\mathfrak{x}_{\mathfrak{t}} + \mathrm{PD}[\widehat{F}] = \mathfrak{y}_{\mathfrak{t}}$. This implies that

$$\mathfrak{m}(\mathfrak{x}_{\mathfrak{t}}) - \mathfrak{m}(\mathfrak{y}_{\mathfrak{t}}) = -n(\ell+1) = -n \pmod{n\ell}.$$

Denote by f_{hor} the sum

$$f_{hor.} = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(Y_n(K))} f_v^{\mathfrak{x}_{\mathfrak{t}}}.T^{\mathfrak{m}(\mathfrak{x}_{\mathfrak{t}})}.$$

Similarly, define $f_{ver.}$ using f_w, or alternatively using f_v and the Spin^c structures $\{\mathfrak{y}_{\mathfrak{t}}\}_{\mathfrak{t}}$. Denote by $f : \mathbb{B}_{m+n} \longrightarrow \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ the sum of these two maps. We would like to study the map f under the identification of $\mathbb{B}_{m+n}(\underline{\mathfrak{t}})$ with the complex $\mathbb{D}_1^s(\underline{\mathfrak{t}})$ given by theorem 2.2.

In fact, in the proof of theorem 2.2 we may use the Heegaard diagram $R_{m,n}$. Then the composition of the map $f_{hor.}$ with Φ_0^{-1} may be described (after composing with another filtered chain homotopy) as the map

$$\begin{aligned} h : \mathbb{D}_1^s &\longrightarrow \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}] \\ h[\mathbf{x}, i, j, k, l] &= [\mathbf{x}, i, j] \otimes T^{-s}, \end{aligned}$$

and the composition of $f_{ver.}$ may be described as the map given by

$$\begin{aligned} v : \mathbb{D}_1^s &\longrightarrow \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}] \\ v[\mathbf{x}, i, j, k, l] &= \tau[\mathbf{x}, l, k] \otimes T^{-s-n}. \end{aligned}$$

Here τ denotes the chain homotopy equivalence from $\mathbb{B}$ to itself which comes from changing the pair of marked points (u,v) to (z,w). In fact defining f_{ver} using f_w and then composing with τ is the same as defining it using f_v and the Spin^c structures $\{\mathfrak{y}_\mathfrak{t}\}_\mathfrak{t}$ over the cobordism W_ℓ.

Let $\overline{f}_\ell$ denote the sum of the two maps

$$h, v : \bigoplus_{-\frac{n(\ell+1)}{2} \leq s < \frac{n(\ell+1)}{2}} \mathbb{D}_1^s \longrightarrow \mathbb{B} \otimes \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}],$$

and denote by $\mathbb{M}(\overline{f}_\ell)$ the mapping cone of $\overline{f}_\ell$. The filtration $\mathcal{F}$ may be extended to a filtration on $\mathbb{M}(\overline{f}_\ell)$ defined by

$$\mathcal{G} : (\text{Generators of } \mathbb{M}(\overline{f}_\ell)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
$$\begin{cases} \mathcal{G}([\mathbf{x}, i, j, k, l]) = (\max(i,l), \max(j,k)) & \text{if } [\mathbf{x}, i, j, k, l] \in \mathbb{D}_1^s, \\ \mathcal{G}([\mathbf{x}, i, j] \otimes T^k) = (i,j) & \text{if } [\mathbf{x}, i, j] \otimes T^k \in \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}]. \end{cases}$$

The complex $\mathbb{M}(\overline{f}_\ell)$ decomposes according to the class of s modulo the integer n and we get

$$\mathbb{M}(\overline{f}_\ell) = \bigoplus_{[s] \in \frac{\mathbb{Z}}{n\mathbb{Z}}} \mathbb{M}^{[s]}(\overline{f}_\ell).$$

We would like to identify the decomposition of $\mathbb{M}^{[s]}(\overline{f}_\ell)$ according to the relative Spin^c structures $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y_n(K), K_n) = \underline{\mathrm{Spin}}^c(Y, K)$. To this end, note that the image of a generator $[(\mathbf{x})_l, i, j] \in \mathbb{B}_n$ under the chain map g is of the form "$[(\mathbf{x})_l, i, j]+$ lower order terms" as an element of $\mathbb{B}_{m+n}$. In particular, the relative Spin^c class of $[\mathbf{x}, i, j, k, l] \in \mathbb{D}_1^s \subset \mathbb{M}^{[s]}(\overline{f}_\ell)$ under the map Φ_0^{-1} should be declared equal to the relative Spin^c class of $[(\mathbf{x})_{k-j}, \max(i,l), \max(j,k)] \in \mathbb{B}_n$, which is already computed to be equal to

$$\underline{\mathfrak{s}}(\mathbf{x}) + (j - k + n(i-j))\mathrm{PD}[\mu].$$

Since $[\mathbf{x}, i, j, k, l] \in \mathbb{D}_1^s$ is mapped to both $[\mathbf{x}, i, j] \otimes T^{-s}$ and to $[\mathbf{x}, l, k] \otimes T^{-s-n}$, it is easy to check that the relative Spin^c structure associated with $[\mathbf{x}, i, j] \otimes T^k \in \mathbb{B}[n\ell]$ should be defined equal to $s(\mathbf{x}) + (-k + n(i-j))\mathrm{PD}[\mu]$. Here we assume that $-\frac{n\ell}{2} \leq k < \frac{n\ell}{2}$. Note that for any generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, by definition we have $s(\mathbf{x}) + i(\underline{\mathfrak{s}}(\mathbf{x}))\mathrm{PD}[\mu] = \underline{\mathfrak{s}}(\mathbf{x})$. It is straight-forward to check that these assignments are respected by the chain maps.

From the above constructions we get a map from $\mathrm{CFK}^\infty(Y_n(K), K_n, \underline{\mathfrak{t}})$ to the complex $\mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}]$; the subset of $\mathbb{M}^{[s]}(\overline{f}_\ell)$ corresponding to the relative Spin^c class $\underline{\mathfrak{t}}$ with $[i(\underline{\mathfrak{t}})] = [s]$ in $\frac{\mathbb{Z}}{n\mathbb{Z}}$. This map respects the filtration on the two sides and it will produce a quasi-isomorphism in homology in the following sense.

Recall that if $f : C \longrightarrow C'$ is a chain map between $\mathbb{Z} \oplus \mathbb{Z}$ chain complexes such that for any positive test domain P (see the introduction for the definition of a positive test domain) then f will induce a chain map $f : C^P \longrightarrow C'^P$ where C^P and C'^P denote the chain complexes generated by the pre-images of P in C and C'. If the induced map

$$f_*^P : H_*^P(C) \longrightarrow H_*^P(C')$$

is an isomorphism for all positive test domains we will call f a *quasi-isomorphism*. The above argument proves that the constructed map

$$CFK^{\infty}(Y_n(K), K_n, \underline{\mathfrak{t}}) \longrightarrow \mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}]$$

is a quasi isomorphism for any $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y_n(K), K_n)$, where $[s] = [i(\underline{\mathfrak{t}})] \in \frac{\mathbb{Z}}{n\mathbb{Z}}$ is the class of $i(\underline{\mathfrak{t}})$ modulo the integer n, and $\mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}]$ is the subcomplex of $\mathbb{M}^{[s]}(\overline{f}_\ell)$ produced by generators in the relative Spin^c class $\underline{\mathfrak{t}}$.

In fact, for any positive test domain P we may follow the above process using complexes determined by the test domain P and the filtrations. Then we will obtain maps

$$(\mathrm{CFK}^{\infty}(Y_n(K), K_n, \underline{\mathfrak{t}})^P \longrightarrow (\mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}])^P$$

which give isomorphisms in homology.

As m goes to infinity we may construct the stabilization of this complex as follows. $\mathbb{D}_1^s(\underline{\mathfrak{t}})$ will be defined as before without any restriction on how big s is. The complex $\mathbb{D}_1(\underline{\mathfrak{t}})$ will decompose as

$$\mathbb{D}_1(\underline{\mathfrak{t}}) = \bigoplus_{\substack{s\in\mathbb{Z}\\ s=i(\underline{\mathfrak{t}}) \pmod n}} \mathbb{D}_1^s(\underline{\mathfrak{t}}).$$

We also introduce a complex $\mathbb{B}[\infty] = \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ as the complex generated by the elements $\{[\mathbf{x}, i, j] \otimes T^k \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i, j, k \in \mathbb{Z}\}$ over $\mathbb{Z}$, with the differential

$$\partial_{\mathbb{B}}[\mathbf{x}, i, j] \otimes T^k = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#(\widehat{\mathcal{M}}(\phi))[\mathbf{y}, i - n_u(\phi), j - n_v(\phi)] \otimes T^k.$$

The maps $\overline{f}_\ell$ stabilize to give a map $\overline{f} : \mathbb{D}_1 \to \mathbb{B}[\infty]$ where $\overline{f} = f + v$ and

$$\begin{aligned} &h[\mathbf{x}, i, j, k, l] = [\mathbf{x}, i, j] \otimes T^{-s},\\ &v[\mathbf{x}, i, j, k, l] = \tau[\mathbf{x}, l, k] \otimes T^{-s-n}, \quad \text{for } [\mathbf{x}, i, j, k, l] \in \mathbb{D}_1^s. \end{aligned}$$

We may take the mapping cone $\mathbb{M}(\overline{f})$ of $\overline{f}$.

The filtration $\mathcal{G}$ on $\mathbb{M}(\overline{f})$ is given by

$$\mathcal{G} : (\text{Generators of } \mathbb{M}(\overline{f})) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$\begin{cases} \mathcal{G}([\mathbf{x}, i, j, k, l]) = (\max(i, l), \max(j, k)) & [\mathbf{x}, i, j, k, l] \in \mathbb{D}_1,\\ \mathcal{G}([\mathbf{x}, i, j] \otimes T^k) = (i, j) & [\mathbf{x}, i, j] \otimes T^k \in \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{\mathbb{Z}}{m\mathbb{Z}}] \end{cases}$$

and the decomposition of $\mathbb{M}(\overline{f})$ according to the relative Spin^c classes is determined by assigning to $a \in \mathbb{M}(\overline{f})$ the relative Spin^c class $\underline{\mathfrak{s}}(a) \in \underline{\mathrm{Spin}}^c(Y_n(K), K_n)$ defined by

$$\underline{\mathfrak{s}}(a) = \begin{cases} \underline{\mathfrak{s}}(\mathbf{x}) + ((j-k) + n(i-j))\mathrm{PD}[\mu] & \text{if } a = [\mathbf{x}, i, j, k, l] \in \mathbb{D}_1\\ s(\mathbf{x}) + (-k + n(i-j))\mathrm{PD}[\mu] & \text{if } a = [\mathbf{x}, i, j] \otimes T^k \in \mathbb{B}[\infty]. \end{cases}$$

The results of the above considerations, together with the argument given in [OS3] for theorem 4.1 will prove the following theorem:

Theorem 3.1. *Suppose that (Y,K) is a null-homologous knot, n is a nonzero integer, and that $(Y_n(K),K_n)$ is the result of n-surgery on (Y,K). Suppose that $\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n)$ is a fixed relative Spin^c class, and that the complex $\mathbb{M}(\overline{f})$ is constructed as above. Then $\mathrm{CFK}^\infty(Y_n(K),K_n,\underline{\mathfrak{t}})$ and $\mathbb{M}(\overline{f})[\underline{\mathfrak{t}}]$ are quasi-isomorphic.*

Proof. The proof is almost identical with the proof of theorem 4.1 in [OS3], which occupies most of that paper. For the reader's convenience, we sketch the proof omitting the details.

The triangle maps of the previous section give rise to a triangle of chain maps

$$
\begin{aligned}
&f_1:\mathrm{CFK}^\infty(Y_n(K),K_n)\longrightarrow \mathrm{CFK}^\infty(Y_{m+n}(K),K_{m+n})\\
&f_2:\mathrm{CFK}^\infty(Y_{m+n}(K),K_{m+n})\longrightarrow \bigoplus^m \mathrm{CFK}^\infty(Y,K)\\
&f_3:\bigoplus^m \mathrm{CFK}^\infty(Y,K)\longrightarrow \mathrm{CFK}^\infty(Y_n(K),K_n)
\end{aligned}
$$

The maps f_1 and f_2 are constructed from the Heegaard diagram $R_{m,n}$ as before. The map f_3 for a generator of the form $[\mathbf{x},i,j]T^k$ is defined by counting triangles $\psi\in\pi_2(\mathbf{x},\Theta,\mathbf{y})$ (where Θ is the canonical generator for the connected sum of $S^1\times S^2$s), such that the number $m_p(\psi)$ is congruent to $c-k$ modulo the integer m, where c is a constant determined from the Heegaard diagram as in [OS3].

Since the compositions $f_2\circ f_2$ and $f_3\circ f_2$ are null-homotopic, f_1 and f_3 induce maps $\Psi:\mathrm{CFK}^\infty(Y_n(K),K_n)\to\mathbb{M}(f_2)$ and $\Psi':\mathbb{M}(f_2)\to\mathrm{CFK}^\infty(Y_n(K),K_n)$, where $\mathbb{M}(f_2)$ denotes the mapping cone of the filtered chain map f_2. These maps are quasi-isomorphisms in the sense discussed earlier (i.e. they induce isomorphisms in homology for any positive test domain P).

Suppose that $m=\ell n$ for some large ℓ. Note that the chain map f_2 is determined as the chain map associated with the 4-manifold cobordism

$$W_\ell: Y_{(\ell+1)n}(K)\longrightarrow Y.$$

For any Spin^c structure $\mathfrak{t}\in\mathrm{Spin}^c(Y_{n(\ell+1)}(K))$ let $\mathfrak{x}_\mathfrak{t},\mathfrak{y}_\mathfrak{t}\in\mathrm{Spin}^c(W_\ell)$ be defined as before. Then if we decompose f_2 as a sum of maps

$$f_2=\sum_{\underline{\mathfrak{t}}\in\mathrm{Spin}^c(W_\ell)} F_{W_\ell,\underline{\mathfrak{t}}}\otimes T^{\mathfrak{m}(\underline{\mathfrak{t}})},$$

and a is a generator in Spin^c class $\mathfrak{t}\in\mathrm{Spin}^c(Y_{n(\ell+1)}(K)$ then the only non-trivial components of $f_2(a)$ in this expression (for large values of ℓ) are the ones of the form $F_{W_\ell,\mathfrak{x}_\mathfrak{t}}(a)$ and $F_{W_\ell,\mathfrak{y}_\mathfrak{t}}(a)$. For any positive test domain P one can check the commutativity of the following diagrams:

$$
\begin{array}{ccc}
\mathrm{CFK}^P(Y_{n(\ell+1)}(K),K_{n(\ell+1)},\underline{\mathfrak{t}}) & \xrightarrow{F_{W_\ell,\mathfrak{x}_\mathfrak{t}}} & (\bigoplus^\ell \mathrm{CFK}^\infty(Y,K))^P\\
\downarrow & & \downarrow\\
\mathbb{D}_1^s[\underline{\mathfrak{t}}]^P & \xrightarrow{\quad h\quad} & (\mathbb{B}[n\ell],\underline{\mathfrak{t}})^P
\end{array}
$$

and the diagram

$$\begin{array}{ccc} \mathrm{CFK}^P(Y_{n(\ell+1)}(K), K_{n(\ell+1)}, \underline{\mathfrak{t}}) & \xrightarrow{F_{W_\ell,\mathfrak{v}_\mathfrak{t}}} & (\bigoplus^\ell \mathrm{CFK}^\infty(Y,K))^P \\ \downarrow & & \downarrow \\ \mathbb{D}_1^s[\underline{\mathfrak{t}}]^P & \xrightarrow{\quad v \quad} & (\mathbb{B}[n\ell], \underline{\mathfrak{t}})^P. \end{array}$$

Here $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)$ is a relative Spin^c structure, $\mathfrak{t}$ is the induced Spin^c structure on $Y_{n(\ell+1)}(K)$ and $-\frac{n(\ell+1)}{2} \leq s < \frac{n(\ell+1)}{2}$ is an integer with the property that $i(t) = s$ modulo the integer $n(\ell+1)$. Furthermore, $(\mathbb{B}[m], \underline{\mathfrak{t}})$ is the part of $\mathbb{B}[m]$ in the relative Spin^c class $\underline{\mathfrak{t}}$.

Note that for a fixed relative Spin^c class $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)$

$$\overline{f}_\ell : \bigoplus_{\substack{-\frac{n(\ell+1)}{2} \leq s < \frac{n(\ell+1)}{2} \\ s = i(\underline{\mathfrak{t}}) (\bmod n)}} \mathbb{D}_1^s(\underline{\mathfrak{t}}) \longrightarrow \bigoplus^{\ell} (\mathbb{B}[m], \underline{\mathfrak{t}})$$

will produce a mapping cone $\mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}]$ which is quasi-isomorphic to the mapping cone of f_2 in the relative Spin^c class $\underline{\mathfrak{t}}$. This may be verified directly from the above commutative diagrams.

For $s \geq \frac{n(\ell+1)}{2}$ note that $i(\underline{\mathfrak{s}}(\mathbf{x})) + j - k = s$ implies that for the generators $[\mathbf{x}, i, j, k, l]$ in $\mathbb{D}_1^s$ we have $j = k + (s - i(\underline{\mathfrak{s}}(\mathbf{x})))$ and $i = l + (s - i(\underline{\mathfrak{s}}(\mathbf{x}))) + 1$. This implies that if ℓ is large enough i is the maximum of $\{i, l\}$ and j is the maximum of $\{j, k\}$. This implies that the map h from $(\mathbb{D}_1^s)^P$ to its image in $\mathbb{B}[\infty]$ is an isomorphism. For such values of s, one may consider $\phi = v \circ h^{-1} : \mathbb{D}_1^s \longrightarrow \mathbb{D}_1^{s+n}$ and observe that $\phi^p([\mathbf{x}, i, j, k, l]$ will have a $\mathbb{Z}^4$-filtration (coming from the integer components) which is less than or equal to

$$\Big[pl - pi + i - \binom{p}{2}, pk - pj + j - \binom{p}{2}, pk - pj + k - \binom{p+1}{2}, pl - pi + i - \binom{p+1}{2}\Big].$$

The positivity of P implies that for fixed $(i,j,k,l) \in \mathbb{Z}^4$ this sequence will eventually leave P.

If $s \leq -\frac{n(\ell+1)}{2}$ a similar argument shows that v from $(\mathbb{D}_1^s)^P$ to its image in $\mathbb{B}[\infty]$ is an isomorphism and $(h \circ v^{-1})^p(a)$ will eventually vanish for a fixed generator a and large enough p. These observations imply that for large values of ℓ, and any positive test domain P there is an isomorphism induced by the inclusion

$$H_*^P(\mathbb{M}^{[s]}(\overline{f}_\ell)[\underline{\mathfrak{t}}]) \longrightarrow H_*^P(\mathbb{M}(\overline{f})[\underline{\mathfrak{t}}]),$$

where $[s]$ denote the class of $i(\underline{\mathfrak{t}})$ modulo the integer n. Thus the chain complex $\mathbb{M}(\overline{f})[\underline{\mathfrak{t}}]$ is quasi-isomorphic to $\mathrm{CFK}^\infty(Y_n(K), K_n, \underline{\mathfrak{t}})$. This would complete the proof. □

Suppose that K is a knot in a homology sphere Y and $s \in \mathbb{Z} = \underline{\mathrm{Spin}}^c(Y,K)$. Denote by $\mathbb{A}_s^+$ the complex generated by those generators $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}^2$ such that $\max(i,j) \geq 0$, and $i(\mathbf{x}) + i - j = s$ as discussed in [OS3], where $i(\mathbf{x}) = i(\underline{\mathfrak{s}}(\mathbf{x}))$. Let $\bar{\mathbb{A}}_s^+$ denote the complex generated by generators $[\mathbf{x}, i] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}^{\geq 0}$ giving $\mathrm{CF}^+(Y)$ for each $s \in \mathbb{Z}$. Let $\mathcal{A}_s^+$ denote the direct sum $\mathcal{A}_s^+ =$

$\bigoplus_{t=s \pmod n} \mathbb{A}_t^+$ and similarly define $\bar{\mathcal{A}}_s^+$. Let $h', v' : \mathcal{A}_s^+ \longrightarrow \bar{\mathcal{A}}_s^+$ be the maps sending $[\mathbf{x}, i, j] \in {\mathbb{A}_t}^+$ to

$$h'[\mathbf{x}, i, j] = [\mathbf{x}, i] \in \bar{\mathbb{A}}_t^+, \quad \text{and}$$
$$v'[\mathbf{x}, i, j] = \tau[\mathbf{x}, j] \in \bar{\mathbb{A}}_{t-n}^+$$

respectively. Denote by ζ_s the sum of these two maps. The above theorem implies the result of [OS3] that the chain homotopy type of $\mathrm{CF}^+(Y_n(K), s)$ is the same as that of the mapping cone of ζ_s:

Corollary 3.2. *Suppose that (Y, K) and $Y_n(K)$ are the same as before. Define $\mathcal{A}_s^+$ and $\bar{\mathcal{A}}_s^+$ and also the map ζ_s as above for $-\frac{n}{2} \leq s < \frac{n}{2}$. Then the homology groups $\mathrm{HF}^+(Y_n(K), \mathfrak{s}_n)$ (where $\mathfrak{s}_n$ is any Spin^c structure with $i_n(\mathfrak{s}_n) = s(mod\ n)$) is the same as the homology of the mapping cone of ζ_s in the Spin^c class $\mathfrak{s}_n$, as computed in* [OS3].

Proof. If the three-manifold Y is a homology sphere and (Y, K) is the given knot, then the set of relative Spin^c structures $\underline{\mathrm{Spin}}^c(Y_n(K), K_n) = \underline{\mathrm{Spin}}^c(Y, K)$ is naturally isomorphic with $\mathbb{Z}$. The map

$$\mathcal{G}_n : \mathbb{Z} = \underline{\mathrm{Spin}}^c(Y_n(K), K_n) \longrightarrow \mathrm{Spin}^c(Y_n(K)) = \frac{\mathbb{Z}}{n\mathbb{Z}}$$

is the reduction modulo the integer n. For $-\frac{n}{2} \leq s < \frac{n}{2}$, thought of as an element of $\underline{\mathrm{Spin}}^c(Y_n(K), K_n)$, the complex $\mathrm{CFK}^\infty(Y_n(K), K_n; s)$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex which is, as a $\mathbb{Z}$ filtered chain complex, chain homotopic with $\mathrm{CF}^\infty(Y_n(K), [s]_{\mathrm{mod}\ n})$. Using theorem 3.1 $C^+(s) = \mathrm{CF}^+(Y_n(K), [s]_{\mathrm{mod}\ n})$ is generated by two types of generators, some of them in $\mathbb{D}_1$ and some of them in $\mathbb{B}[\infty]$.

If $[\mathbf{x}, i, j, k, l] \in \mathbb{D}_1$ is a generator of $C^+(s)$ it is implied that $i(\mathbf{x}) + (j - k) + n(i - j) = s$. This implies that if $i(\mathbf{x}) + j - k$ is congruent to s modulo n, then $i = -\frac{i(\mathbf{x})+j-k-s}{n} + j$ will compute i. Note that from the relation $i - j + k - l = 1$, the integer l may be computed as well. The filtration $\mathcal{G}$ gives the value $(\max(i, l), \max(j, k))$ on this generator. As a result the projection over the second component of the $\mathbb{Z} \oplus \mathbb{Z}$ filtration will be $\max(j, k)$. Let $\mathbb{A}_t^+$ for $t = s \pmod n$ denote the complex generated by those $[\mathbf{x}, j, k] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}^2$ such that $i(\mathbf{x}) + j - k = t$ and $\max(j, k) \geq 0$. There is a natural correspondence between the generators of $C^+(s)$ of the form $[\mathbf{x}, i, j, k, l] \in \mathbb{D}_1$ with the generators of $\bigoplus_{t=s \pmod n} \mathbb{A}_t^+$.

If $[\mathbf{x}, i, j] \otimes T^k \in \mathbb{B}[\infty]$ is a generator of $C^+(s)$ it is implied that $-k + n(i - j) = s$. As a result, $-k = s \pmod n$, and the value of i is determined once such a choice for k is fixed. Thus, the generators of the form $[\mathbf{x}, i, j] \otimes T^{-t}$ for $t = s \pmod n$ are in correspondence with generators of $\bar{\mathbb{A}}_t^+$. This complex is generated by $[\mathbf{x}, i] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}$ with $i \geq 0$, which gives $\mathrm{CF}^+(Y)$. $\bar{\mathbb{A}}_t^+$ is the same complex for all values of t.

Under the above correspondence the map h takes $[\mathbf{x}, j, k] \in \mathbb{A}_t^+$ to $[\mathbf{x}, j] \in \bar{\mathbb{A}}_t^+$, and the map v takes the same generator to $\tau[\mathbf{x}, k] \in \bar{\mathbb{A}}_{t-n}^+$, where τ is the chain homotopy equivalence discussed earlier. The homology of the mapping cone $C^+(s)$ is thus computed exactly as proved in [OS3].

□

Finally note that in the case of null-homologous knots, there is a natural isomorphism of chain complexes (without the filtration) from $\mathbb{D}_1$ to $\mathbb{B}[\infty]$. Namely, we may define:

$$\Psi : \mathbb{D}_1 \longrightarrow \mathbb{B}[\infty]$$
$$\Psi[\mathbf{x}, i, j, k, l] := [\mathbf{x}, i, j] \otimes T^{-i(\underline{\mathfrak{s}}(\mathbf{x}))+k-j}.$$

It is easy to verify that this in fact is an isomorphism. The map $\overline{f}$ will introduce a map $\overline{g}_n : \mathbb{D}^{up} \longrightarrow \mathbb{D}^{down}$, where $\mathbb{D}^{up}$ and $\mathbb{D}^{down}$ are copies of $\mathbb{D}_1$. The map induced from h is the identity. The map v_n induced by v is more interesting:

$$v_n[\mathbf{x}, i, j, k, l] = \tau[\mathbf{x}, l, k, 2k - j - n, 2l - i - n].$$

Note that the chain homotopy equivalence on $\mathbb{B}$ may be naturally extended to a chain homotopy equivalence from the complex $\mathbb{D}_1$ (equipped with the $\mathbb{Z} \oplus \mathbb{Z}$ filtration coming from projection over the 3rd and 4th integer components of the generators) to $\mathbb{D}^{down}$. There is a discussion on this in the introduction. We denote this new chain homotopy equivalence by the same letter τ.

The filtration induced on $\mathbb{D}^{down}$ from $\mathbb{B}[\infty]$ is given by the first two integer components of the generators. We obtain the following re-statement of theorem 3.1:

Theorem 3.3. *Let $(Y, K), n$ and $\mathbb{D}_\delta$ be as above. Let $\mathbb{D}^{up}$ and $\mathbb{D}^{down}$ be copies of the complex $\mathbb{D}_1$. Denote by $v_n : \mathbb{D}^{up} \longrightarrow \mathbb{D}^{down}$ the map defined by $v_n[\mathbf{x}, i, j, k, l] = \tau[\mathbf{x}, l, k, 2k - j - n, 2l - i - n]$. Let $\overline{g}_n = Id + v_n$ and denote by $\mathbb{M}(\overline{g}_n)$ the mapping cone of $\overline{g}_n$. Define a filtration $\mathcal{G}$ on $\mathbb{M}(\overline{g}_n)$ by setting*

$$\mathcal{G}[\mathbf{x}, i, j, k, l] = \begin{cases} (max(i, l), max(j, k)) & \text{if } [\mathbf{x}, i, j, k, l] \in \mathbb{D}^{up}, \\ (i, j) & \text{if } [\mathbf{x}, i, j, k, l] \in \mathbb{D}^{down}. \end{cases}$$

Also define a map from the set of generators to the set of relative Spin^c *structures* $\underline{\mathrm{Spin}}^c(Y_n(K), K_n) = \underline{\mathrm{Spin}}^c(Y, K)$ *by*

$$\underline{\mathfrak{s}}[\mathbf{x}, i, j, k, l] = \underline{\mathfrak{s}}(\mathbf{x}) + ((j - k) + n(i - j))\mathrm{PD}[\mu]$$
$$\text{for any generator } [\mathbf{x}, i, j, k, l] \in \mathbb{D}^{up} \text{ or } \mathbb{D}^{down},$$

and split $\mathbb{M}(\overline{g}_n)$ as $\mathbb{M}(\overline{g}_n) = \bigoplus_{\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)} \mathbb{M}(\overline{g}_n)[\underline{\mathfrak{t}}]$. Then $\mathrm{CFK}^\infty(Y_n(K), K_n, \underline{\mathfrak{t}})$ is quasi-isomorphic to $\mathbb{M}(\overline{g}_n)[\underline{\mathfrak{t}}]$ for every $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y, K)$.

It will be more convenient to state our next result as a generalization of the first form of this theorem. The theorem look nicer, however, in this second form when we work with a null-homologous knot.

4. Rationally null-homologous knots

In this section we generalize the construction of previous section to the case of rationally null-homologous knots (Y, K). We remind the reader of a couple of facts from [OS4] where the notion of knot Floer homology is generalized to the case of rationally null-homologous knots and also the integral surgery formulas are generalized to Morse surgery formulas for this type of knots.

Note that to a rationally null-homologous knot (Y, K) is associated a notion of relative Spin^c structure and the set of such structures is denoted by $\underline{\mathrm{Spin}}^c(Y, K)$. There is a surjective reduction map

$$G_{Y,K} : \underline{\mathrm{Spin}}^c(Y, K) \longrightarrow \mathrm{Spin}^c(Y).$$

If $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ is a Heegaard diagram for (Y, K) as before, there is a map associated with the marked point p which assigns relative Spin^c structures to intersections of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. Namely we have the map

$$\underline{\mathfrak{s}} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\mathrm{Spin}}^c(Y, K).$$

The choice of a framing λ determines a push-off K_λ of the knot K into the knot complement $Y \setminus \mathrm{nd}(K)$, which provides us, via Poincaré duality, with a cohomology class

$$\mathrm{PD}[\lambda] := \mathrm{PD}[K_\lambda] \in H^2(Y \setminus \mathrm{nd}(K), \partial(Y \setminus \mathrm{nd}(K)), \mathbb{Z}).$$

Note that the set of relative Spin^c structures is an affine space over this later cohomology group.

If λ is a framing for (Y, K), then $\lambda + n\mu$ is also a framing, and we may define the push-off $K_{\lambda+n\mu}$ similar to K_λ. We may also define the cohomology class $\mathrm{PD}[\lambda + n\mu] = \mathrm{PD}[\lambda] + n\mathrm{PD}[\mu]$ similarly.

We may use the framing in place of the curve λ in previous sections to define the curves λ_n, and the knots $(Y_n(K), K_n)$. Again the set of relative Spin^c structures associated with a knot (Y, K) is the same as the set of relative Spin^c structures associated with $(Y_n(K), K_n)$:

$$\underline{\mathrm{Spin}}^c(Y, K) \cong \underline{\mathrm{Spin}}^c(Y_n(K), K_n).$$

Fixing the framing λ we may start the process of second and third sections. Suppose that a Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ for (Y, K) is given as above, inducing a differential ∂^∞ on $\mathrm{CFK}^\infty(Y, K)$, which is generated by $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$. Let $\mathbb{D}$ be the complex generated by $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}^4$ with the differential

$$\partial_{\mathbb{D}}[\mathbf{x}, i, j, k, l] = \sum_p n_p[\mathbf{y}_p, i - i_p, j - i_p, k - k_p, l - k_p],$$

where $\partial^\infty[\mathbf{x}, i, k] = \sum_p n_p[\mathbf{y}_p, i - i_p, k - k_p]$. For $\delta \in \mathbb{Z}$ let $\mathbb{D}_\delta$ denote the subcomplex of $\mathbb{D}$ generated by the generators $a = [\mathbf{x}, i, j, k, l]$ such that $\Delta(a) = i - j + k - l = \delta$.

In [OS4] a map $\Xi : \mathrm{Spin}^c(Y_n(K)) \longrightarrow \underline{\mathrm{Spin}}^c(Y_n(K), K_n)$ is constructed (in the presence of the framing λ and for large values of n) which plays the role of the map sending $\mathfrak{t} \in \mathrm{Spin}^c(Y_n(K))$ to $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y_n(K), K_n)$ such that $s_n(\underline{\mathfrak{t}}) = \mathfrak{t}$ and $-\frac{n}{2} \leq i(\underline{\mathfrak{t}}) < \frac{n}{2}$.

Similar to the definition of $\mathbb{D}_1^{\mathfrak{s}}(\underline{\mathfrak{t}})$ for any relative Spin^c class $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y_n(K), K_n)$ and any Spin^c class $\mathfrak{t}_n \in \mathrm{Spin}^c(Y_n(K))$ denote by $\mathbb{D}_1^{\mathfrak{t}_n}(\underline{\mathfrak{t}})$ the subcomplex of $\mathbb{D}_1$ generated by the generators $[\mathbf{x}, i, j, k, l] \in \mathbb{D}_1$ satisfying

$$\begin{cases} \underline{\mathfrak{s}}(\mathbf{x}) + (j - k)\mathrm{PD}[\mu] + (i - j)\mathrm{PD}[\lambda + n\mu] = \underline{\mathfrak{t}} \\ \underline{\mathfrak{s}}(\mathbf{x}) + (j - k)\mathrm{PD}[\mu] = \Xi(\mathfrak{t}_n). \end{cases}$$

It is implied that if $\mathbb{D}_1^{\mathfrak{t}_n}(\underline{\mathfrak{t}})$ is non-empty then $G_n(\underline{\mathfrak{t}}) = \mathfrak{t}_n$.

The complexes $\mathbb{D}_1^{\mathfrak{t}_n}(\underline{\mathfrak{t}})$ are the natural replacements for $\mathbb{D}_1^{s}(\underline{\mathfrak{t}})$, and we may follow the process used for proving theorem 2.2 to prove the following:

Theorem 4.1. *Suppose that (Y,K) is a rationally null-homologous knot, λ is a framing for K and $(Y_n(K),K_n)$ is as above. Construct the complex $\mathbb{D}_1$ as before Then for large values of $n\in\mathbb{Z}$ the filtered chain complex associated with the rationally null-homologous knot $(Y_n(K),K_n)$ in relative* Spin^c *class $\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n)$ has the same chain homotopy type as the complex $\mathbb{D}_1^{\mathfrak{t}_n}(\underline{\mathfrak{t}})$ equipped with the $\mathbb{Z}\oplus\mathbb{Z}$ filtration given by*

$$\mathcal{F}[\mathbf{x},i,j,k,l] = (max\{i,l\},\ max\{j,k\}).$$

Here we have chosen $\mathfrak{t}_n$ so that $\mathfrak{t}_n = G_n(\underline{\mathfrak{t}})$.

Proof. All the steps in the proof are completely similar to the steps in the proof for the null-homologous case. ☐

Define the complex $\mathbb{B}[\infty]$ as the complex generated by the generators $[\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}}$ where $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $i,j\in\mathbb{Z}$ and $\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y,K)$ and we have the relation

$$\mathfrak{s}(\mathbf{x}) = G_{Y,K}(\underline{\mathfrak{t}}) = G(\underline{\mathfrak{t}})$$

Clearly this is a generalization of the definition of the complex $\mathbb{B}[\infty]$ used in the third section.

We may construct two maps from the complex $\mathbb{D}_1$ to $\mathbb{B}[\infty]$ as follows. These maps will be given via the formulas

$$\begin{aligned}
&h, v_\lambda : \mathbb{D}_1 \longrightarrow \mathbb{B}[\infty],\\
&h[\mathbf{x},i,j,k,l] = [\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{s}}(\mathbf{x})+(j-k)\mathrm{PD}[\mu]}\\
&v_\lambda[\mathbf{x},i,j,k,l] = \tau[\mathbf{x},l,k]\otimes T^{\underline{\mathfrak{s}}(\mathbf{x})+(j-k)\mathrm{PD}[\mu]+\mathrm{PD}[\lambda]}.
\end{aligned}$$

Define $f_\lambda = h+v_\lambda$ and let $\mathbb{M}(f_\lambda)$ denote the mapping cone of f_λ. Define a filtration $\mathcal{G}$ on the generators of $\mathbb{M}(f_\lambda)$ by

$$\begin{cases}\mathcal{G}([\mathbf{x},i,j,k,l]) = (\max(i,l),\max(j,k)) & [\mathbf{x},i,j,k]\in\mathbb{D}_1\\ \mathcal{G}([\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}}) = (i,j) & [\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}}\in\mathbb{B}[\infty].\end{cases}$$

The relative Spin^c classes of generators in $\mathbb{M}(f_\lambda)$ will be defined via

$$\underline{\mathfrak{s}}(a) = \begin{cases}\underline{\mathfrak{s}}(\mathbf{x})+(j-k)\mathrm{PD}[\mu]+(i-j)\mathrm{PD}[\lambda] & \text{if } a=[\mathbf{x},i,j,k,l]\in\mathbb{D}_1,\\ \underline{\mathfrak{t}}+(i-j)\mathrm{PD}[\lambda] & \text{if } a=[\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}}\in\mathbb{B}[\infty].\end{cases}$$

We may insert these constructions, which are the generalized versions of the previous ones, in the proof of theorem 3.1 to obtain the following.

Theorem 4.2. *Suppose that (Y,K) is a rationally null-homologous knot, λ is a framing for K, and that $(Y_\lambda(K),K_\lambda)$ is the knot obtained as above by Morse surgery with framing λ on (Y,K). Suppose that the complex $\mathbb{M}(f_\lambda)$ be as above, and let $\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n)$ be a relative* Spin^c *structure. Then the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain*

complex $\mathrm{CFK}^\infty(Y_\lambda(K),K_\lambda,\underline{\mathfrak{t}})$ *is quasi-isomorphic to the complex* $\mathbb{M}(f_\lambda)[\underline{\mathfrak{t}}]$ *associated with the mapping cone of the map* $f_\lambda = h + v_\lambda : \mathbb{D}_1 \longrightarrow \mathbb{B}[\infty]$*, equipped with the* $\mathbb{Z}\oplus\mathbb{Z}$*-filtration* $\mathcal{G}$*.*

Proof. Again, all the necessary modifications are minor. The proof of theorem 6.1 in [OS4] may be combined with our techniques for the proof of theorem 3.1 to prove the above theorem. □

Remark 4.3. *This is a generalization of theorem 6.1 from* [OS4] *in an obvious way.*

5. Rational surgeries on null-homologous knots

The natural application of the above generalization of theorem 3.1 is a computation for null-homologous knots of the homologies of the Heegaard Floer complex associated with rational surgeries on them.

For simplicity, we choose to deal with the case where Y is a homology sphere, so that the knot (Y,K) is automatically null-homologous. As in [OS4], if $\frac{p}{q}\in\mathbb{Q}$ is a rational number, then write

$$\frac{p}{q}=\frac{r}{q}+a=\frac{r}{q}+\lfloor\frac{p}{q}\rfloor,$$

and note that $(Y_{\frac{p}{q}}(K),K_{\frac{p}{q}})$ may be obtained by a Morse surgery with coefficient a on the knot $K\#O_{\frac{q}{r}}\subset Y\#L(q,r)$. We remind the reader that $O=O_{\frac{q}{r}}$ is the knot obtained as one component of the Hopf link in the three-manifold $L=L(q,r)$ obtained by a $\frac{q}{r}$ surgery on the second component of the Hopf link. Note that (L,O) is a U-knot, according to [OS4].

We remind the reader of a number of facts from [OS4] about the splitting of relative Spin^c structures under connected sum of knots and about the filtered chain homotopy type of $(Y_1\#Y_2,K_1\#K_2)$ (in terms of the chain homotopy type of (Y_1,K_1)) when (Y_2,K_2) is a U-knot. First note that there is a connected sum map

$$\underline{\mathrm{Spin}}^c(Y_1,K_1)\times\underline{\mathrm{Spin}}^c(Y_2,K_2)\longrightarrow\underline{\mathrm{Spin}}^c(Y_1\#Y_2,K_1\#K_2)$$

sending a pair of relative Spin^c structures $(\underline{\mathfrak{s}}_1,\underline{\mathfrak{s}}_2)$ to $\underline{\mathfrak{s}}_1\#\underline{\mathfrak{s}}_2$. Note that there is a one-parameter family of pairs $(\underline{\mathfrak{s}}_1,\underline{\mathfrak{s}}_2)$ such that $\underline{\mathfrak{s}}_1\#\underline{\mathfrak{s}}_2$ is a fixed relative Spin^c class in $\underline{\mathrm{Spin}}^c(Y_1\#Y_2,K_1\#K_2)$. If K_2 is a U-knot for any $\underline{\mathfrak{s}}_1\in\underline{\mathrm{Spin}}^c(Y_1,K_1)$ and any $\mathfrak{s}_2\in\mathrm{Spin}^c(Y_2)$ there exists a unique relative Spin^c class $\underline{\mathfrak{s}}_2\in\underline{\mathrm{Spin}}^c(Y_2,K_2)$ with the property that $G_{Y_2,K_2}(\underline{\mathfrak{s}}_2)=\mathfrak{s}_2$ such that there is an equivalence of chain homotopy types

$$\mathrm{CFK}^\infty(Y_1,K_1,\underline{\mathfrak{s}}_1)\cong\mathrm{CFK}^\infty(Y_1\#Y_2,K_1\#K_2,\underline{\mathfrak{s}}_1\#\underline{\mathfrak{s}}_2).$$

In particular for the knot $(L(q,r),O_{\frac{q}{r}})$ there is a commutative diagram

$$\begin{array}{ccc}\mathbb{Z} & \xrightarrow{\ \phi\ } & \underline{\mathrm{Spin}}^c(L(q,r),O_{\frac{q}{r}})\\ \downarrow & & \downarrow {\scriptstyle G_{L,O}}\\ \frac{\mathbb{Z}}{q\mathbb{Z}} & \xrightarrow{\ \cong\ } & \mathrm{Spin}^c(L(q,r))\end{array}$$

such that for $0 \le i \le q-1$ there is an isomorphism $\widehat{\mathrm{HFK}}(L(q,r), O_{\frac{q}{r}}; \phi(i)) \cong \mathbb{Z}$, and such that for all other i we have $\widehat{\mathrm{HFK}}(L(q,r), O_{\frac{q}{r}}, \phi(i)) = 0$.

For any homology sphere Y and any knot (Y,K), we may note that

$$\underline{\mathrm{Spin}^c}(Y,K) \cong \underline{\mathrm{Spin}^c}(Y\#L, K\#O) \cong \mathbb{Z} \text{ and}$$
$$H^2(Y,K) \cong H^2(L,O) \cong H^2(Y\#L, K\#O) \cong \mathbb{Z}.$$

Under these isomorphisms the following diagram is commutative

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\ \downarrow & & \downarrow \\ H^2(Y,K)\oplus H^2(L,O) & \longrightarrow & H^2(Y\#L, K\#O) \end{array}$$

where f is defined via $f(x,y) = qx+y$. Suppose that K_λ is the push-off of the knot $K\#O$ with respect to the framing a (where $a + \frac{r}{q} = \frac{p}{q}$) into the complement of this knot in $Y\#L$. Then according to [OS4] the Poincaré dual PD$[\lambda]$ of the homology class represented by K_λ represents the element

$$p \in \mathbb{Z} = H^2(Y\#L, K\#O) = H^2(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}}).$$

The meridian of the knot $K\#O$ in $Y\#L$ is just the image of the meridian μ of the knot (Y,K) in the connected sum. As a result, the push-off K_μ is obtained as the image of the push-off of the curve μ (in the complement of K in Y) under the map f constructed above. Using the above isomorphisms, this corresponds to the element

$$q \in \mathbb{Z} = H^2(Y\#L, K\#O) = H^2(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}}).$$

We need to construct the complex $\mathbb{D}_1$ out of the complex $\mathbb{D}$ associated with $\mathrm{CFK}^\infty(Y\#L, K\#O)$. To this end note that this complex in the relative Spin^c class $s \in \mathbb{Z} = \underline{\mathrm{Spin}^c}(Y\#L, K\#O)$ corresponds to the complex $\mathrm{CFK}^\infty(Y,K)$ in the relative Spin^c class $[\frac{s}{q}] \in \mathbb{Z} = \underline{\mathrm{Spin}^c}(Y,K)$. More precisely there is a filtered quasi-isomorphism between the two complexes.

Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ be a pointed Heegaard diagram for the knot (Y,K) and assume that the complex $\mathrm{CFK}^\infty(Y,K)$ is generated by the generators $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$. According to the above paragraph, the complex $\mathrm{CFK}^\infty(Y\#L, K\#O)$ is generated by the generators of the form $[\mathbf{x}, i, j] \otimes \zeta^t$ where $0 \le t < q$, $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i, j \in \mathbb{Z}$. This implies that the corresponding complex $\mathbb{D}'$ associated with $\mathrm{CFK}^\infty(Y\#L, K\#O)$ is of the form $\mathbb{D} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[\zeta]}{\zeta^q=1}$, where $\mathbb{D}$ is associated with the Heegaard diagram H as before. Similarly $\mathbb{D}_1' = \mathbb{D}_1 \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[\zeta]}{\zeta^q=1}$ may be obtained. The $\mathbb{Z}\oplus\mathbb{Z}$ filtration on $\mathbb{D}_1'$ comes from the $\mathbb{Z}\oplus\mathbb{Z}$ filtration on the $\mathbb{D}_1$ factor according to the above construction. The relative Spin^c structures associated with the generators of $\mathbb{D}_1'$ are described as follows. If $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a generator and if $i(\mathbf{x}) \in \mathbb{Z} = \underline{\mathrm{Spin}^c}(Y,K)$ is the relative Spin^c structure associated with it then the relative Spin^c structure in $\mathbb{Z} = \underline{\mathrm{Spin}^c}(Y\#L, K\#O) = \underline{\mathrm{Spin}^c}(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}})$ associated with $\mathbf{x} \otimes \zeta^t$ under the above correspondence will be $qi(\mathbf{x}) + t$. The computation of relative Spin^c structures in theorem 4.2 then implies that the relative Spin^c class associated with a generator $[\mathbf{x}, i, j, k, l] \otimes \zeta^t$ is given by the following formula

$$i([\mathbf{x}, i, j, k, l] \otimes \zeta^t) = qi(\mathbf{x}) + p(i-j) + q(j-k) + t \in \mathbb{Z} \cong \underline{\mathrm{Spin}^c}(Y_{\frac{p}{q}}(K), K_{\frac{p}{q}}).$$

Note that $\mathbb{B}[\infty]$ is generated by generators $[\mathbf{x},i,j]\otimes\zeta^t\otimes T^{\underline{\mathfrak{t}}}$ such that $qi(\mathbf{x})+t=\underline{\mathfrak{t}}$. As a result, the value of t is determined from $\underline{\mathfrak{t}}\in\mathbb{Z}\cong\underline{\mathrm{Spin}}^c(Y_{\frac{p}{q}},K_{\frac{p}{q}})$. Thus the complex $\mathbb{B}^\infty$ is in fact generated by the generators of the form $[\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}}$ where $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $i,j\in\mathbb{Z}$ and $\underline{\mathfrak{t}}\in\mathbb{Z}\cong\underline{\mathrm{Spin}}^c(Y_{\frac{p}{q}},K_{\frac{p}{q}})$. The relative Spin^c structure in $\underline{\mathrm{Spin}}^c(Y_{\frac{p}{q}},K_{\frac{p}{q}})$ associated with any such generator is given, according to theorem 4.2, by the following formula

$$i([\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}})=\underline{\mathfrak{t}}+p(i-j)\in\mathbb{Z}\cong\underline{\mathrm{Spin}}^c(Y_{\frac{p}{q}},K_{\frac{p}{q}}).$$

The maps from $\mathbb{D}_1\otimes_{\mathbb{Z}}\frac{\mathbb{Z}[\zeta]}{\zeta^q=1}$ to $\mathbb{B}[\infty]$ are given by

$$\begin{aligned}h([\mathbf{x},i,j,k,l]\otimes\zeta^t)&=[\mathbf{x},i,j]\otimes T^{q(i(\mathbf{x})+j-k)+t}\\ v([\mathbf{x},i,j,k,l]\otimes\zeta^t)&=[\mathbf{x},l,k]\otimes T^{q(i(\mathbf{x})+j-k)+t+p}.\end{aligned}$$

Let $\mathbb{M}(\overline{f})$ denote the mapping cone of $\overline{f}=h+v$. Define a $\mathbb{Z}\oplus\mathbb{Z}$ grading on the generators of $\mathbb{M}(\overline{f})$ by

$$\begin{aligned}\mathcal{G}([\mathbf{x},i,j,k,l]\otimes\zeta^t)&=(max(i,l),max(j,k))\\ \mathcal{G}([\mathbf{x},i,j]\otimes T^{\underline{\mathfrak{t}}})&=(i,j).\end{aligned}$$

This complex (and consequently its homology) is decomposed into a direct sum according to the relative Spin^c structures:

$$\mathbb{M}(\overline{f})=\bigoplus_{\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y,K)}\mathbb{M}(\overline{f})[\underline{\mathfrak{t}}].$$

Although the filtered chain homotopy type may change in the course of this process (as a quasi-isomorphism is composed with a chain homotopy equivalence), the homology is preserved.

Theorem 5.1. *Let Y be a homology sphere and let (Y,K) denote a knot in K. Suppose that $\frac{p}{q}>0$ is a rational number and let $(Y_{\frac{p}{q}},K_{\frac{p}{q}})$, $\overline{f}$ and $\mathbb{M}(\overline{f})$ be as before. Then for any relative* Spin^c *class $\underline{\mathfrak{t}}\in\underline{\mathrm{Spin}}^c(Y,K)$, the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex* $\mathrm{CFK}^\infty(Y_{\frac{p}{q}}(K),K_{\frac{p}{q}},\underline{\mathfrak{t}})$ *is quasi-isomorphic to the mapping cone $\mathbb{M}(\overline{f})[\underline{\mathfrak{t}}]$.*

Note that this theorem may be re-stated as in the introduction.

6. Non-vanishing results for $\widehat{\mathrm{HFK}}(K_{\frac{p}{q}})$

In this section we consider the special case where $P=\{(0,0)\}$ and K is a knot in S^3 where the construction simplifies significantly. A non-vanishing result may be proved for rational surgeries on K which may be used for re-proving *Property P* as discussed in the introduction.

We will use the rational surgery formula as stated in the introduction. Suppose that $\frac{p}{q}$ is a positive rational number and let $L=K_{\frac{p}{q}}$ denote the result of $\frac{p}{q}$ surgery on K. For simplicity, we will denote $\mathbb{D}^{up}$ by $\mathbb{A}$ and $\mathbb{D}^{down}$ by $\mathbb{B}$. For this particular choice of P, we will denote $\mathbb{A}^P$ by $\widehat{\mathbb{A}}$ and $\mathbb{B}^P$ by $\widehat{\mathbb{B}}$. The map $f=I+g_{\frac{p}{q}}$ from $\mathbb{A}$ to $\mathbb{B}$ induces a chain map from $\widehat{\mathbb{A}}$ to $\widehat{\mathbb{B}}$ and $\widehat{\mathrm{HFK}}(L)$ is in fact the homology of the complex

$$H_*(\widehat{\mathbb{A}})\xrightarrow{f_*}H_*(\widehat{\mathbb{B}}).$$

Note that $\widehat{\mathbb{B}}(\underline{\mathfrak{t}})$ for any relative Spin^c structure

$$\underline{\mathfrak{t}} \in \mathbb{Z} \cong \underline{\mathrm{Spin}}^c(S^3, K)$$

is generated by $[\mathbf{x}, 0, 0, k, k-1] \otimes T^t$ such that $q(i(\mathbf{x}) - k) + t = \underline{\mathfrak{t}}$. This implies that $i(\mathbf{x}) - k = \underline{\mathfrak{s}} = \lfloor \underline{\mathfrak{t}}/q \rfloor$ and $t = q\{\underline{\mathfrak{t}}/q\}$. The homology of this complex is just $\widehat{\mathrm{HF}}(S^3, \mathfrak{s}_0) = \mathbb{Z}$, where $\mathfrak{s}_0$ is the unique Spin^c structure on S^3.

The complex $\widehat{\mathbb{A}}$ is more interesting. Consider a generator $a = [\mathbf{x}, i, j, k, l] \otimes T^t$ in $\widehat{\mathbb{A}}(\underline{\mathfrak{t}})$. Suppose that $i = j + \delta$. It is implied that $l = k - 1 + \delta$. As a result

$$\max(j, k) = 0 = \max(i, l) = \delta + \max(j, k-1).$$

This can happen only if $\delta \in \{0, 1\}$. Correspondingly we may write $\widehat{\mathbb{A}} = \widehat{\mathbb{A}}_0 \oplus \widehat{\mathbb{A}}_1$, where $\widehat{\mathbb{A}}_i$ is the part of $\widehat{\mathbb{A}}$ generated by generators as above such that $\delta = i$.

If a is in $\widehat{\mathbb{A}}_0$ then $i = j = 0$ and $k \leq 0$. As a result the complex $\widehat{\mathbb{A}}_0$ may be identified by the complex $\widehat{\mathbb{B}}\{k \leq 0\}$ consisting of the part of complex $\widehat{\mathbb{B}}$ with non-positive k-component. In fact $\widehat{\mathbb{A}}_0(\underline{\mathfrak{t}}) = \widehat{\mathbb{B}}\{k \leq 0\}(\underline{\mathfrak{t}})$. If $C = \mathrm{CFK}^\infty(K)$ denotes the complex generated by $[\mathbf{x}, i, k]$ then one can check that in fact

$$\widehat{\mathbb{A}}_0(\underline{\mathfrak{t}}) \cong \widehat{\mathbb{B}}\{k \leq 0\}(\underline{\mathfrak{t}}) \cong C\{i = 0, k \leq 0\}(\lfloor \frac{\underline{\mathfrak{t}}}{q} \rfloor).$$

The map f_* will be the map induced in homology by the inclusion of $\widehat{\mathbb{A}}_0$ in $\widehat{\mathbb{B}}$.

However, if a is in $\widehat{\mathbb{A}}_1$, then $k = l = 0$ and $i \leq 0$. We will also have

$$q(i(\mathbf{x}) + i - 1) + p + t = \underline{\mathfrak{t}}.$$

Thus, $t = \{\frac{\underline{\mathfrak{t}} - p}{q}\}$ and $i(\mathbf{x}) + i - 1 = \lfloor \frac{\underline{\mathfrak{t}} - p}{q} \rfloor = \underline{\mathfrak{s}}'$. The complex may be identified with $C\{i \leq 0, k = 0\}(\underline{\mathfrak{s}}')$.

We will determine the maximum and minimum values for $\underline{\mathfrak{t}}$ such that the knot Floer homology is non-trivial, i.e. $\widehat{\mathrm{HFK}}(L, \underline{\mathfrak{t}}) \neq 0$.

Theorem 6.1. *Suppose that K is a knot in S^3 of genus $g(K)$, and let $r = \frac{p}{q} \in \mathbb{Q}$ be a positive rational number. Under the natural identification $\underline{\mathrm{Spin}}^c(S^3, K_r) = \mathbb{Z}$ we will have*

$$\widehat{\mathrm{HFK}}(K_r, -qg(K)) \cong \widehat{\mathrm{HFK}}(K_r, qg(K) + p - 1) \cong \widehat{\mathrm{HFK}}(K, g(K)) \neq 0,$$

and for any $\underline{\mathfrak{t}} \in \mathbb{Z}$ such that $\underline{\mathfrak{t}} < -qg(K)$ or $\underline{\mathfrak{t}} \geq qg(K) + p$ we will have

$$\widehat{\mathrm{HFK}}(K_r, \underline{\mathfrak{t}}) = 0.$$

Proof. Suppose that $\underline{\mathfrak{t}} = -qg(K)$. Then $\widehat{\mathbb{A}}_1(\underline{\mathfrak{t}})$ is identified with the complex $C\{i \leq 0, k = 0\}(\underline{\mathfrak{s}}')$ where $\underline{\mathfrak{s}}' = \lfloor \frac{-qg(K) - p}{q} \rfloor < -g(K)$. This implies that if $\mathbf{x}$ is a generator (intersection of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$) such that no generator $[\mathbf{x}, i, 0]$ is included in $C\{i \leq 0, k = 0\}(\underline{\mathfrak{s}}')$ then $i(\mathbf{x}) < \underline{\mathfrak{s}}' < -g(K)$. Such generators will cancel each-other in homology (as $\widehat{\mathrm{HFK}}(K, \underline{\mathfrak{s}}) = 0$ for $\underline{\mathfrak{s}} < -g(K)$, see [OS5]). It is implied that the map

$$f_* : H_*(\widehat{\mathbb{A}}_1(\underline{\mathfrak{t}})) \longrightarrow H_*(\widehat{\mathbb{B}}(\underline{\mathfrak{t}}))$$

is an isomorphism. As a result we will have

$$\widehat{\mathrm{HFK}}(L,-qg(K)) \cong H_*(\widehat{\mathbb{A}}_0(-qg(K))).$$

But $H_*(\widehat{\mathbb{A}}_0(-qg(K))$ is isomorphic to $H_*(C\{i=0,k\leq 0\}(-g(K)))$. If $i(x)-k=-g(K)$ then either $i(\mathbf{x})=-g(K)$ and $k=0$, or $i(\mathbf{x})<-g(K)$. The generators of the later form will disappear in homology by the same reasoning. This implies that

$$\widehat{\mathrm{HFK}}(L,-qg(K)) \cong \widehat{\mathrm{HFK}}(K,-g(K)) \neq 0,$$

where the last non-vanishing result is borrowed from [OS5].

It is clear that if $\underline{\mathfrak{t}} < -qg(K)$ then the first isomorphism may still be constructed. In the second part it is always implied that $i(\mathbf{x})<-g(K)$, thus the homology group $H_*(\widehat{\mathbb{A}}_0(\underline{\mathfrak{t}}))$ is trivial.

Now assume that $\underline{\mathfrak{t}} = qg(K)+p-1$. This time $\widehat{\mathbb{A}}_0(\underline{\mathfrak{t}})$ may be identified with $C\{i=0,k\leq 0\}(\underline{\mathfrak{s}})$ where $\underline{\mathfrak{s}}=\lfloor\frac{\underline{\mathfrak{t}}}{q}\rfloor \geq g(K)$. For any $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ with $i(\mathbf{x})\leq g(K)$ one can find a non-positive integer k such that $i(\mathbf{x})-k=\underline{\mathfrak{s}}$ (i.e. $[\mathbf{x},0,k]\in C\{i=0,k\leq 0\}(\underline{\mathfrak{s}})$). As before, this implies that

$$f_*: H_*(\widehat{\mathbb{A}}_0(\underline{\mathfrak{t}})) \longrightarrow H_*(\widehat{\mathbb{B}}(\underline{\mathfrak{t}}))=\mathbb{Z}$$

is an isomorphism and $\widehat{\mathrm{HFK}}(L,qg(K)+p-1)\cong H_*(\widehat{\mathbb{A}}_1(qg(K)+p-1))$. Note that for this value of $\underline{\mathfrak{t}}$ we have

$$\underline{\mathfrak{s}}' = \lfloor\frac{\underline{\mathfrak{t}}-p}{q}\rfloor = \lfloor\frac{qg(K)-1}{q}\rfloor = g(K)-1.$$

The generators $[\mathbf{x},i,0]$ of $C\{i\leq 0,k=0\}(\underline{\mathfrak{s}}')$ should then satisfy $i(\mathbf{x})+i-1=g(K)-1$. This equality implies that $i(\mathbf{x})\geq g(K)$. The generators with $i(\mathbf{x})>g(K)$ are killed in homology. What remains is the set of generators $[\mathbf{x},0,0]$ such that $i(\mathbf{x})=g(K)$ which shows that

$$\widehat{\mathrm{HFK}}(L,qg(K)+p-1)\cong\widehat{\mathrm{HFK}}(K,g(K))\neq 0,$$

where again we use the result of [OS5] for the last part. It is clear from the above argument that for $\underline{\mathfrak{t}}\geq qg(K)+p$ the knot Floer homology groups will vanish. □

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